Media IC and System Lab
Graduate Institute of Electronics Engineering
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Computer Vision: Geometry 101

2024 Crash Course

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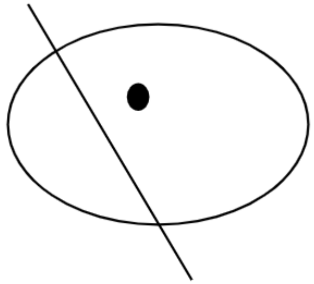
Outline

- **1. Title**
- **2. Projective 2D Geometry**
- **3. Estimation of Transformations**
- **4. Pinhole Camera Model**
- **5. Ending**

Projective 2D Geometry

- Points, lines & Conics



- Transformations & Invariants

Homogeneous Coordinates

- Homogeneous representation of lines

$$ax + by + c = 0 \qquad \underline{(a,b,c)^{\mathrm{T}}}$$

$$(ka)x + (kb)y + kc = 0, \forall k \neq 0 \qquad (a,b,c)^{\mathrm{T}} \sim k(a,b,c)^{\mathrm{T}}$$

equivalence class of vectors, any vector is representative

- Homogeneous representation of points

$\mathrm{x} = (x, y)^{\mathrm{T}}$ on $\underline{\mathrm{l}} = (a,b,c)^{\mathrm{T}}$ if and only if $ax + by + c = 0$

$$(x,y,1)(a,b,c)^{\mathrm{T}} = (x,y,1)\underline{\mathrm{l}} = 0 \qquad (x, y, 1)^{\mathrm{T}} \sim k(x, y, 1)^{\mathrm{T}}, \forall k \neq 0$$

The point x lies on the line l if and only if $\mathrm{x}^{\mathrm{T}}\underline{\mathrm{l}}=\underline{\mathrm{l}}^{\mathrm{T}}\mathrm{x}=0$

Degrees of Freedom

Homogeneous coordinates $(x_1, x_2, x_3)^{\mathrm{T}}$ but only 2DOF

Inhomogeneous coordinates $(x, y)^{\mathrm{T}}$

The point x=$(x_1, x_2, x_3)^{\mathrm{T}}$ represent the point $(x_1/x_3, x_2/x_3)^{\mathrm{T}}$ in $\mathbb{R}^2$

ex: $(2,3,1)^T = (4,6,\underline{2})^T = (6,9,\underline{3})^T$

Points and Lines

- Homogeneous Coordinate

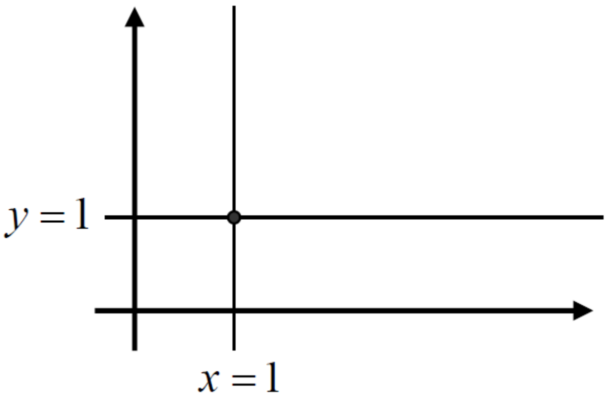
$$\boldsymbol{p} = \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \tilde{\boldsymbol{p}} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad \boldsymbol{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \tilde{\boldsymbol{p}} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- Intersections of lines
  - The intersection of two lines l and l’ is **x = l x l’**
- Line joining two points
  - The line through two points x and x’ is **l = x x x’**

Points and Lines (cont.)

- Exercise

  - *Example 1*



  - *Example 2*

    – Determine the line passing through the two points (2,4) and (5,13)

Points and Lines (cont.)

- Solution
  - *Example 2*

Determine the line passing through the two points (2,4) and (5,13)

1.Homogeneous representation of points

$$\tilde{\boldsymbol{x}}_1 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \in \mathbb{P}^2 \qquad \tilde{\boldsymbol{x}}_2 = \begin{bmatrix} 5 \\ 13 \\ 1 \end{bmatrix} \in \mathbb{P}^2$$

2.Homogeneous representation of lines

$$\tilde{\boldsymbol{l}} = \tilde{\boldsymbol{x}}_1 \times \tilde{\boldsymbol{x}}_2 = [\tilde{\boldsymbol{x}}_1]_\times \tilde{\boldsymbol{x}}_2 = \begin{bmatrix} 0 & -1 & 4 \\ 1 & 0 & -2 \\ -4 & 2 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 13 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

3.Equation of the line

$$-3x + y + 2 = 0 \quad \Leftrightarrow \quad y = 3x - 2$$

Matrix representation of the cross product

$$\boldsymbol{u} \times \boldsymbol{v} \mapsto [\boldsymbol{u}]_\times \boldsymbol{v}$$

where

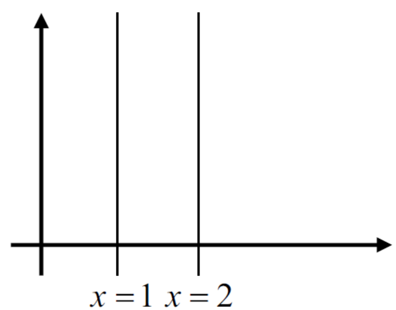
$$[\boldsymbol{u}]_\times \stackrel{def}{=} \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

Ideal Points and the Line at Infinity

- Intersections of parallel lines

$$\mathbf{l} = (a, b, c)^{\mathsf{T}} \text{ and } \mathbf{l}' = (a, b, c')^{\mathsf{T}} \qquad \mathbf{l} \times \mathbf{l}' = (b, -a, 0)^{\mathsf{T}}$$

Example



$(b, -a)$ tangent vector (line's direction)

(a, b) normal direction

Ideal points $(x_1, x_2, 0)^{\mathsf{T}}$

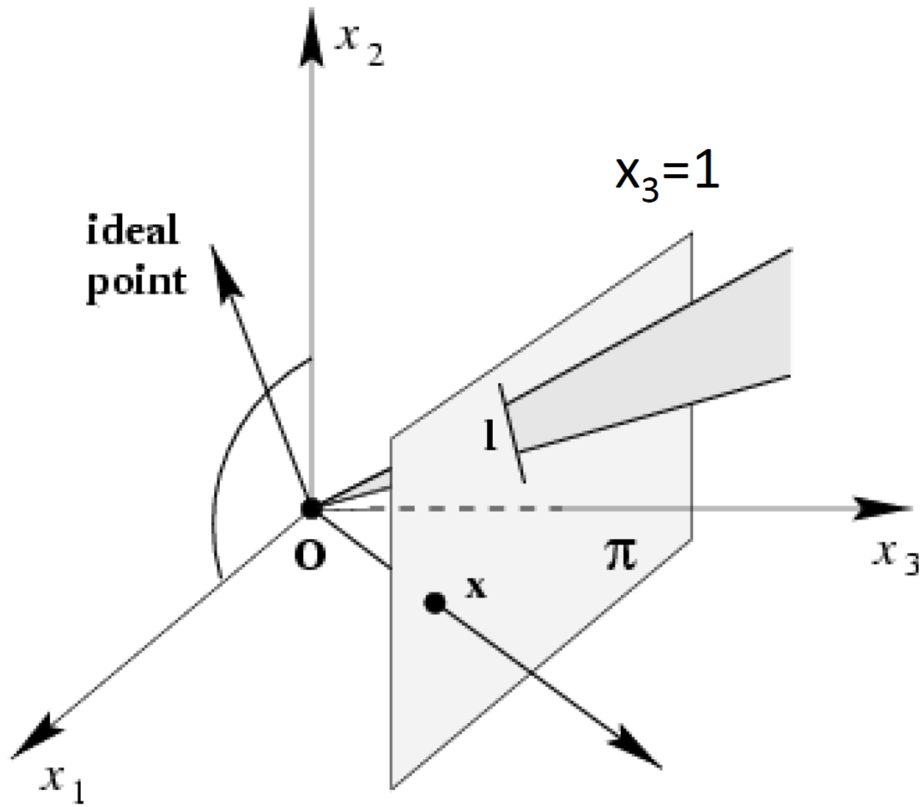
Line at infinity $\mathbf{l}_\infty = (0, 0, 1)^{\mathsf{T}}$

$$\mathbf{P}^2 = \mathbf{R}^2 \cup \mathbf{l}_\infty$$

Note that in $\mathbf{P}^2$ there is no distinction between ideal points and others

Projection 座標系 = 二維座標系 + Line at Infinity

A Model for the Projective Plane



Projective Transformations

- Definition

A *projectivity* is an invertible mapping h from P^2 to itself such that three points x_1,x_2,x_3 lie on the same line if and only if $h(x_1),h(x_2),h(x_3)$ do.

- Theorem

A mapping $h:P^2\rightarrow P^2$ is a projectivity if and only if there exist a non-singular 3x3 matrix **H** such that for any point in P^2 reprented by a vector x it is true that $h(x)=\mathbf{H}x$

- Definition: Projective Transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad \mathbf{x}' = \mathbf{H}\,\mathbf{x}$$

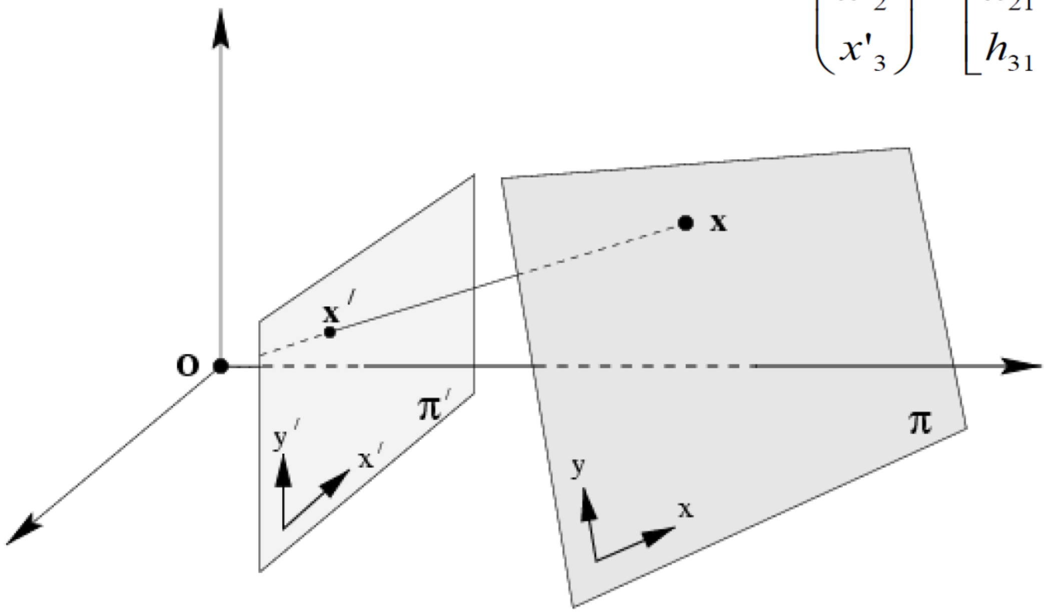
8DOF

projectivity=collineation=projective transformation=homography

Mapping between Planes

- 2D → 2D

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$



central projection may be expressed by x'=Hx

(application of theorem)

Removing Projective Distortion

select four points in a plane with know coordinates

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

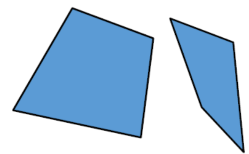
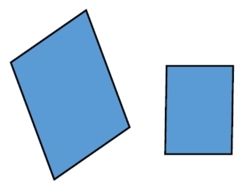
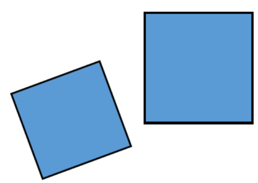
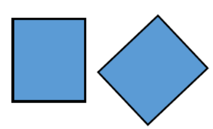
$$x'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{11}x + h_{12}y + h_{13}$$
$$y'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{21}x + h_{22}y + h_{23}$$

(linear in h_{ij})

(2 constraints/point, 8DOF $\Rightarrow$ 4 points needed)

Remark: no calibration at all necessary

Summary of Transformations

			Invariant Properties
Projective 8dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio
Affine 6dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallellism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids). **The line at infinity l_∞**
Similarity 4dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratios of lengths, angles. **The circular points I,J**
Euclidean 3dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		lengths, areas.

Summary of Transformations (cont.)

- Class I: Isometries

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \varepsilon\cos\theta & -\sin\theta & t_x \\ \varepsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \varepsilon = \pm 1$$

orientation preserving: $\varepsilon = 1$ **(Euclidean transform)**
orientation reversing: $\varepsilon = -1$

$$\mathrm{x}' = \mathbf{H}_E\,\mathrm{x} = \begin{bmatrix} \mathbf{R} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix} \mathrm{x} \qquad \mathbf{R}^{\mathsf{T}}\mathbf{R} = \mathbf{I}$$

3DOF (1 rotation, 2 translation), can be computed from 2 point correspondences
special cases: pure rotation, pure translation

Invariants: length, angle, area

- Class II: Similarities

$$\mathrm{x}' = \mathbf{H}_S\,\mathrm{x} = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix} \mathrm{x} \qquad \mathbf{R}^{\mathsf{T}}\mathbf{R} = \mathbf{I}$$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

4DOF (1 scale, 1 rotation, 2 translation),
can be computed from 2 point correspondences

also know as *equi-form* (shape preserving)

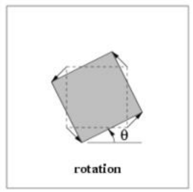
Invariants: ratios of length, angle, ratios of areas, parallel lines

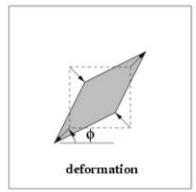
Summary of Transformations (cont.)

- Class III: Affine Transformations

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\mathrm{x}' = \mathbf{H}_A\, \mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix} \mathrm{x}$$





6DOF (2 scale, 2 rotation, 2 translation),
can be computed from 3 point correspondences
non-isotropic scaling! (2DOF: scale ratio and orientation)

Invariants: parallel lines, ratios of parallel lengths, ratios of areas

$$\mathbf{A} = \mathbf{R}(\theta)\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi) \qquad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

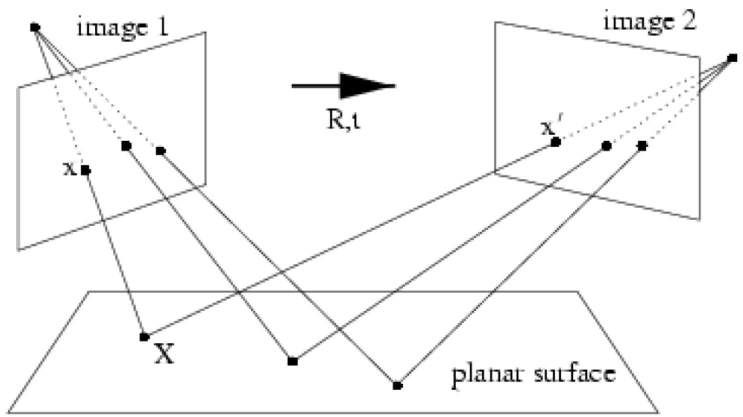
- **Class IV: Projective Transformations** ★

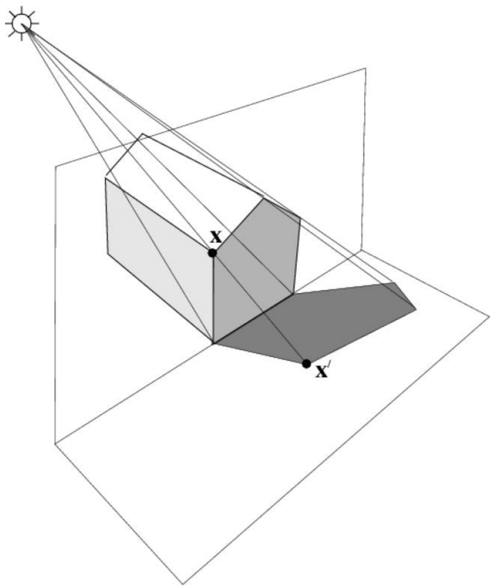
$$\mathrm{x}' = \mathbf{H}_P\, \mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix} \mathrm{x} \qquad \mathrm{v} = (v_1, v_2)^{\mathsf{T}}$$

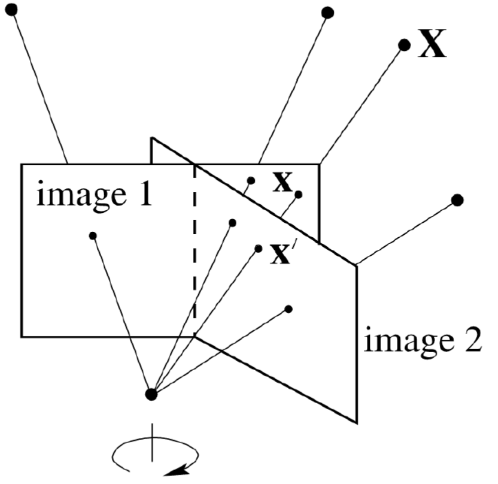
8DOF (2 scale, 2 rotation, 2 translation, 2 line at infinity)
can be computed from 4 point correspondences
Action non-homogeneous over the plane

Invariants: cross-ratio of four points on a line, (ratio of ratio)

Application







2D Homography

- Given a set of (Xi, Xi'), compute H (Xi = HXi)

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\lambda \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2 independent equations / point
8 degrees of freedom

  - Minimal Solution: 4 points yield an exact solution for H
    - No exact solution (noise)

Gold Standard Algorithm

- Direct Linear Transformation (DLT)

$$\mathrm{x}'_i = \mathrm{H}\mathrm{x}_i \rightarrow \underline{\mathrm{x}'_i \times \mathrm{H}\mathrm{x}_i = 0}$$

$$\mathrm{x}'_i \times \mathrm{H}\mathrm{x}_i = \begin{pmatrix} y'_i \mathrm{h}^{3\top}\mathrm{x}_i - w'_i \mathrm{h}^{2\top}\mathrm{x}_i \\ w'_i \mathrm{h}^{1\top}\mathrm{x}_i - x'_i \mathrm{h}^{3\top}\mathrm{x}_i \\ x'_i \mathrm{h}^{2\top}\mathrm{x}_i - y'_i \mathrm{h}^{1\top}\mathrm{x}_i \end{pmatrix}$$

$$\begin{bmatrix} 0^\top & -w'_i \mathrm{x}_i^\top & y'_i \mathrm{x}_i^\top \\ w'_i \mathrm{x}_i^\top & 0^\top & -x'_i \mathrm{x}_i^\top \\ -y'_i \mathrm{x}_i^\top & x'_i \mathrm{x}_i^\top & 0^\top \end{bmatrix} \begin{pmatrix} \mathrm{h}^1 \\ \mathrm{h}^2 \\ \mathrm{h}^3 \end{pmatrix} = 0$$

$$\mathrm{A}_i \mathrm{h} = 0$$

Direct Linear Transformation

$$\mathrm{x}'_i = \mathrm{H}\mathrm{x}_i \rightarrow \underline{\mathrm{x}'_i \times \mathrm{H}\mathrm{x}_i = 0}$$

- Only 2 out of 3 equations are linearly independent

$$\begin{bmatrix} 0^\top & -w'_i\mathrm{x}_i^\top & y'_i\mathrm{x}_i^\top \\ w'_i\mathrm{x}_i^\top & 0^\top & -x'_i\mathrm{x}_i^\top \\ -y'_i\mathrm{x}_i^\top & x'_i\mathrm{x}_i^\top & 0^\top \end{bmatrix} \begin{pmatrix} \mathrm{h}^1 \\ \mathrm{h}^2 \\ \mathrm{h}^3 \end{pmatrix} = 0 \Rightarrow \begin{bmatrix} 0^\top & -w'_i\mathrm{x}_i^\top & y'_i\mathrm{x}_i^\top \\ w'_i\mathrm{x}_i^\top & 0^\top & -x'_i\mathrm{x}_i^\top \end{bmatrix} \begin{pmatrix} \mathrm{h}^1 \\ \mathrm{h}^2 \\ \mathrm{h}^3 \end{pmatrix} = 0$$

- Solving for H

$$\mathrm{Ah} = 0 \qquad \begin{bmatrix} \mathrm{A}_1 \\ \mathrm{A}_2 \\ \vdots \\ \mathrm{A}_n \end{bmatrix} \mathrm{h} = 0$$

- Additional constraint: $||h|| = 1$
- $Ah = 0$ not possible, so minimize $||Ah||$

Direct Linear Transformation (cont.)

- DLT Algorithm

<u>Objective</u>

Given $n \geq 4$ 2D to 2D point correspondences $\{x_i \leftrightarrow x_i'\}$, determine the 2D homography matrix H such that $x_i' = Hx_i$

<u>Algorithm</u>

(i) For each correspondence $x_i \leftrightarrow x_i'$ compute A_i. Usually only the first two rows are needed.

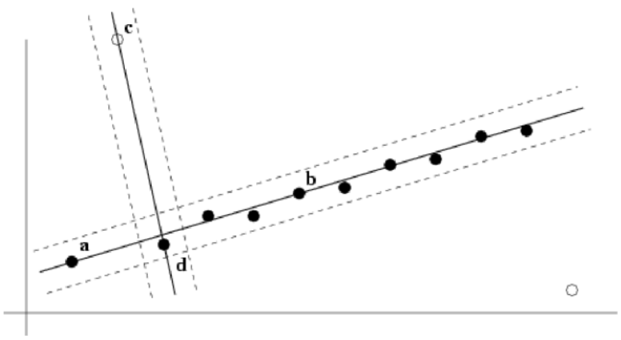
(ii) Assemble n 2x9 matrices A_i into a single $2n$x9 matrix A

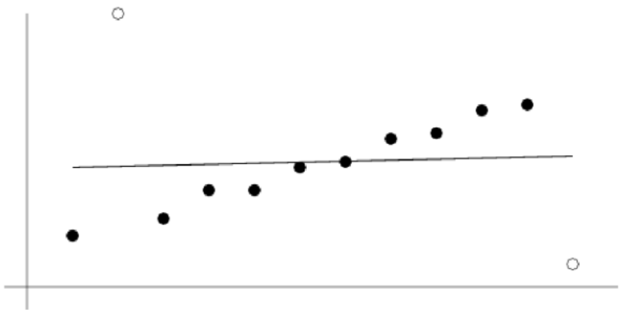
(iii) Obtain SVD of A. Solution for h is the last column of V

(iv) Determine H from h

Robust Estimation

- Gross outliers?





Robust Estimation (cont.)

- RANSAC: RANdom SAmple Consensus

<u>Objective</u>

Robust fit of model to data set S which contains outliers

<u>Algorithm</u>

(i) Randomly select a sample of *s* data points from S and instantiate the model from this subset.

(ii) Determine the set of data points S_i which are within a distance threshold *t* of the model. The set S_i is the consensus set of samples and defines the inliers of S.

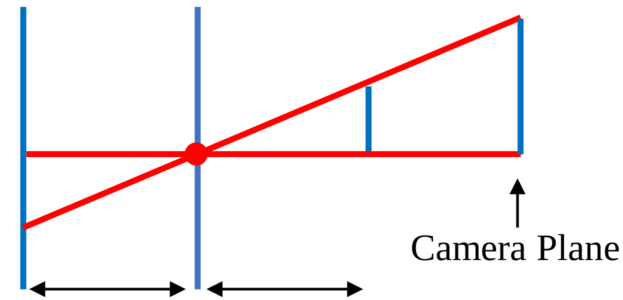
(iii) If the subset of S_i is greater than some threshold *T*, re-estimate the model using all the points in S_i and terminate

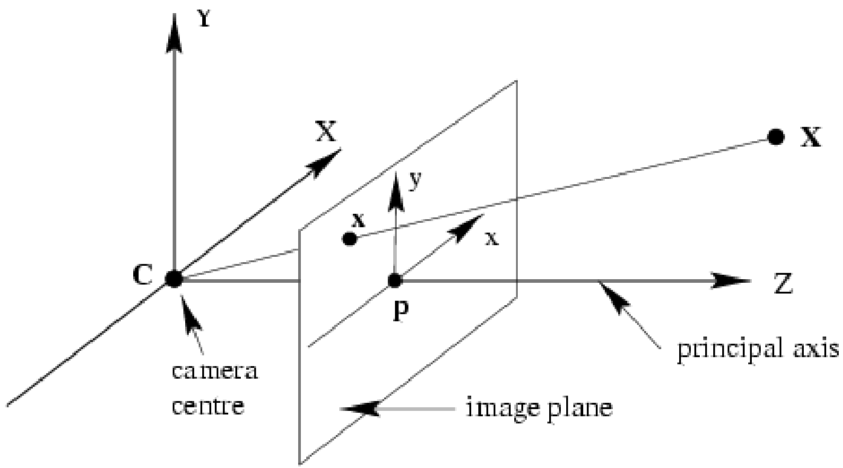
(iv) If the size of S_i is less than *T*, select a new subset and repeat the above.

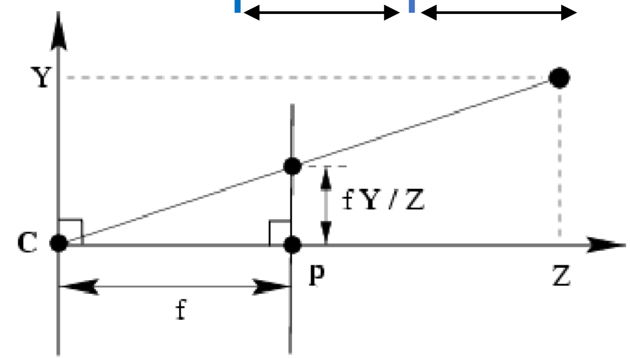
(v) After *N* trials the largest consensus set S_i is selected, and the model is re-estimated using all the points in the subset S_i

Camera Models

- Pinhole Camera Model







$$(X,Y,Z)^T \mapsto \underline{(fX/Z, fY/Z)^T}$$

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \underline{\begin{pmatrix} fX \\ fY \\ Z \end{pmatrix}} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Pinhole Camera Model

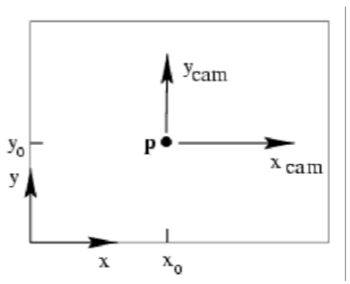
$$(X,Y,Z)^T \mapsto \underline{(fX/Z, fY/Z)^T}$$

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \underline{\begin{pmatrix} fX \\ fY \\ Z \end{pmatrix}} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\boxed{\begin{gathered} \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \underline{\begin{bmatrix} f & & \\ & f & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ & 1 & & 0 \\ & & 1 & 0 \end{bmatrix}} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \\ \mathrm{x} = \underline{\mathrm{P}}\mathrm{X} \\ \mathrm{P} = \mathrm{diag}(f, f, 1)[\mathrm{I} \mid 0] \end{gathered}}$$

Pinhole Camera Model (cont.)

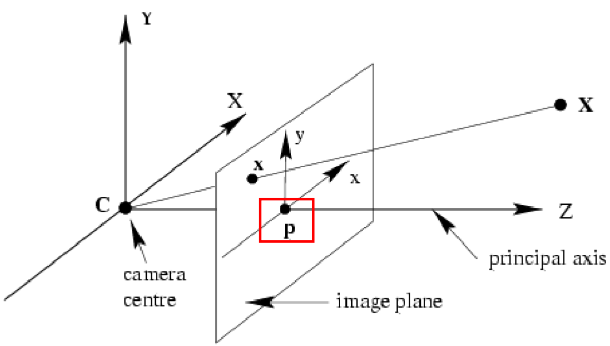
- Principal Point Offset



(0,0,0,1)

$\xrightarrow{?}$ (0,0,1)

$\rightarrow$ (Px,Py,1)



$$(X,Y,Z)^T \mapsto \underline{(fX/Z, fY/Z)^T}$$

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \underline{\begin{pmatrix} fX \\ fY \\ Z \end{pmatrix}} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$(X,Y,Z)^T \mapsto (fX/Z + \underline{p_x}, fY/Z + \underline{p_y})^T$$

$(p_x, p_y)^T$ principal point

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX + Zp_x \\ fY + Zp_x \\ Z \end{pmatrix} = \begin{bmatrix} f & & p_x & 0 \\ & f & p_y & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

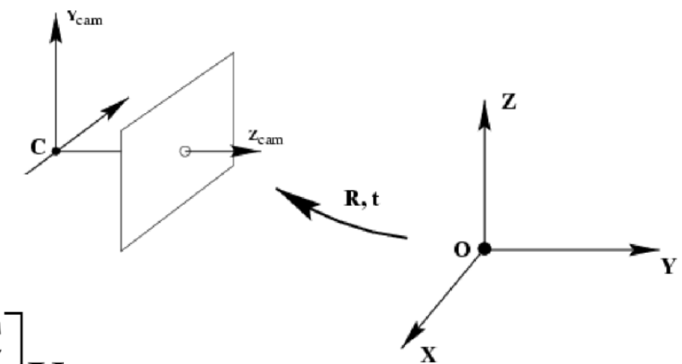
Pinhole Camera Model (cont.)

- Calibration Matrix

$$K = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$

Calibration Matrix

- Rotation and Translation



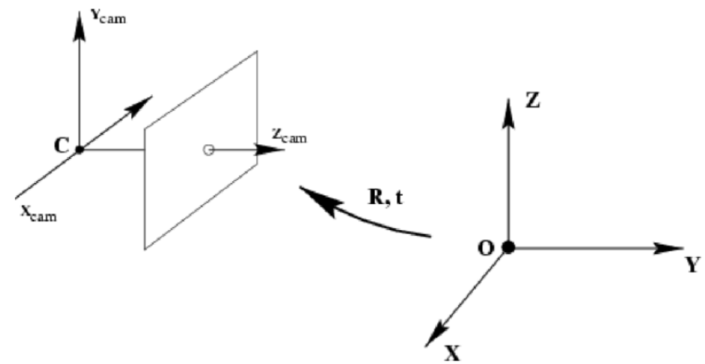
$$\tilde{X}_{cam} = R\left(\tilde{X} - \tilde{C}\right)$$

$$X_{cam} = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} X$$

$$x = K[I \mid 0] X_{cam}$$

Pinhole Camera Model (cont.)

- Rotation and Translation



$$\tilde{X}_{cam} = R\left(\tilde{X} - \tilde{C}\right)$$

$$X_{cam} = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} X$$

$$x = K[I \mid 0]X_{cam}$$

$$x = KR\left[I \mid -\tilde{C}\right]X$$

$$x = PX$$

可被校正

$$P = K[R \mid t] \qquad t = -R\tilde{C}$$

旋轉、平移

Internal Camera Parameter
Internal Orientation
Intrinsic Matrix

External Camera Parameter
Exterior Orientation
Extrinsic Matrix

Pinhole Camera Model (cont.)

- CCD (Charge-Coupled Device) Camera

$$K = \begin{bmatrix} m_x & & \\ & m_y & \\ & & 1 \end{bmatrix} \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} \alpha_x & & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix}$$

- Finite Projective Camera

$$K = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_x & p_y \\ & & 1 \end{bmatrix}$$

s: skew parameter,
=0 for most normal cameras

Congratulations!