Camera Models

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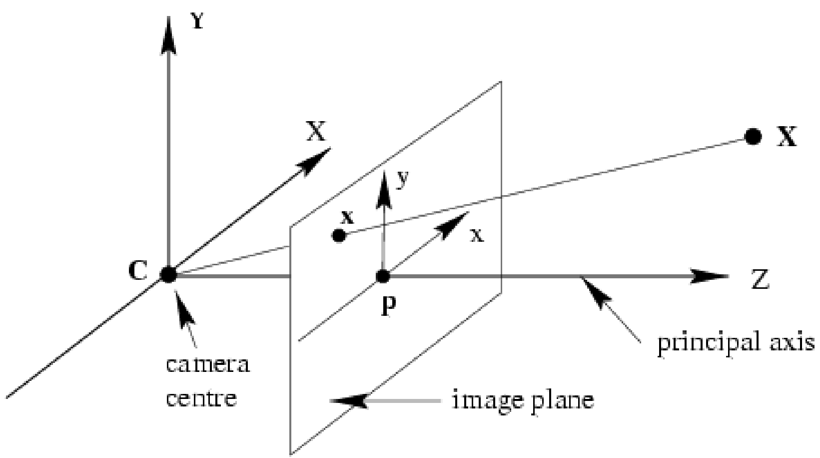
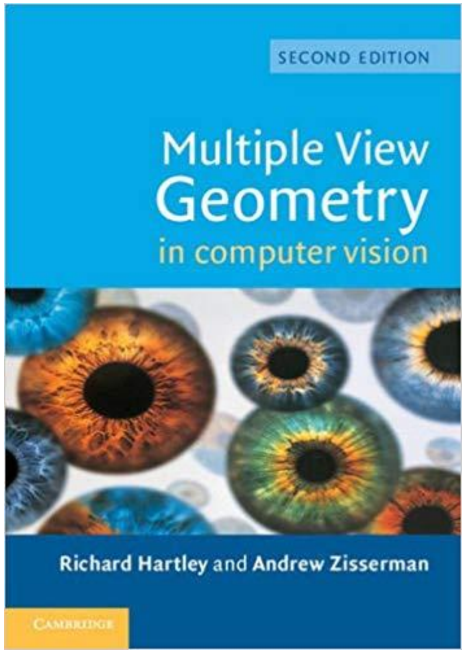
Outline

- Camera models





[Slides credit: Marc Pollefeys]

Camera Obscura: the Pre-Camera

- First idea: Mo-Ti, China (470BC to 390BC)
- First built: Alhazen, Iraq/Egypt (965 to 1039AD)

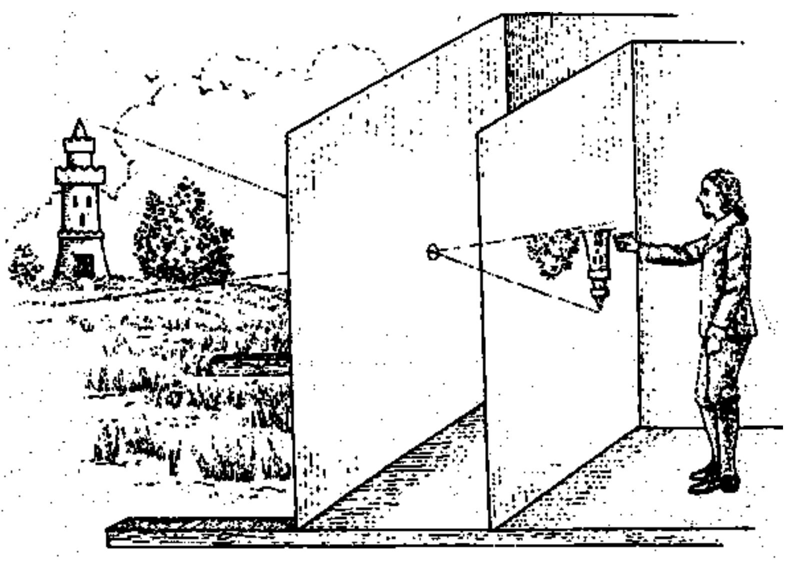


Illustration of Camera Obscura

Freestanding camera obscura at UNC Chapel Hill

Photo by Seth Ilys

Camera Obscura

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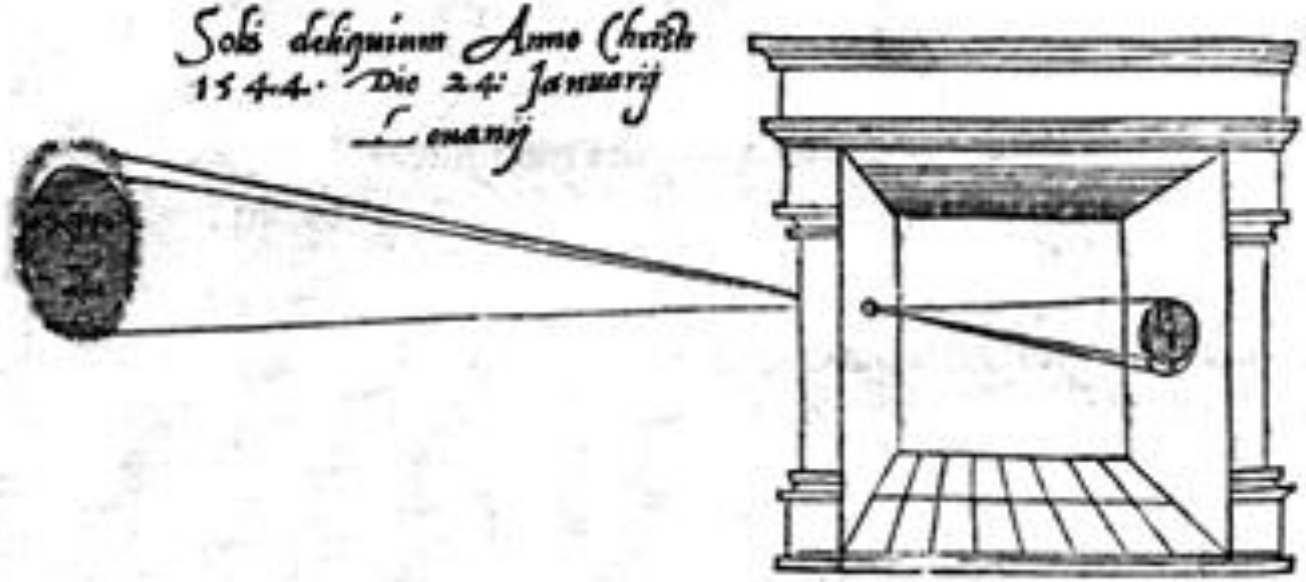

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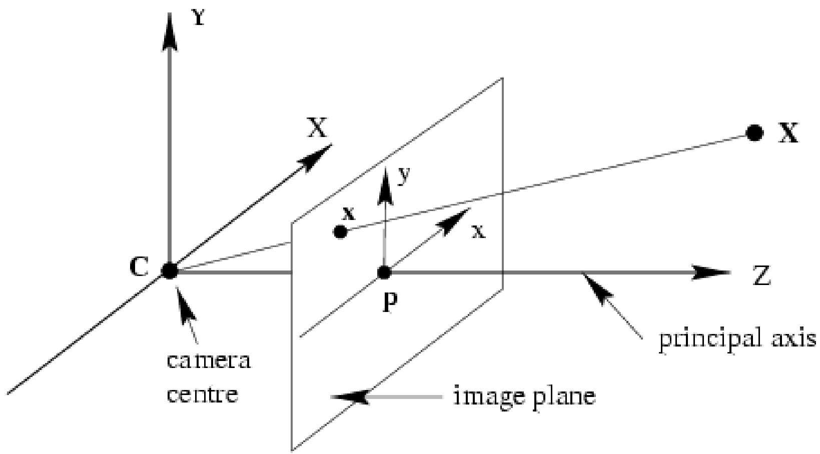
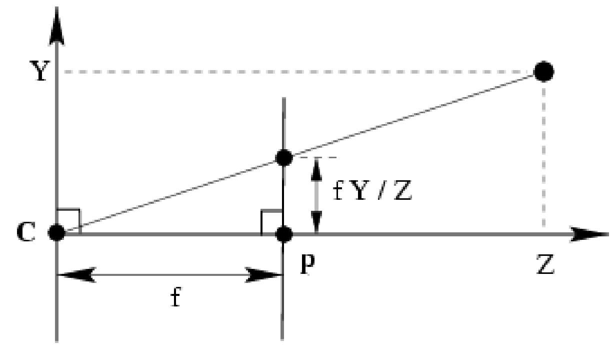
Camera Obscura, Reinerus Gemma-Frisius, 1544

Pinhole Camera Model

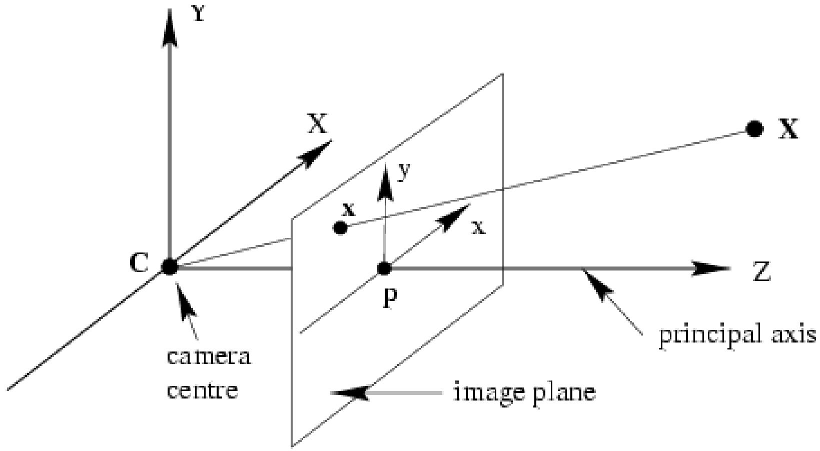


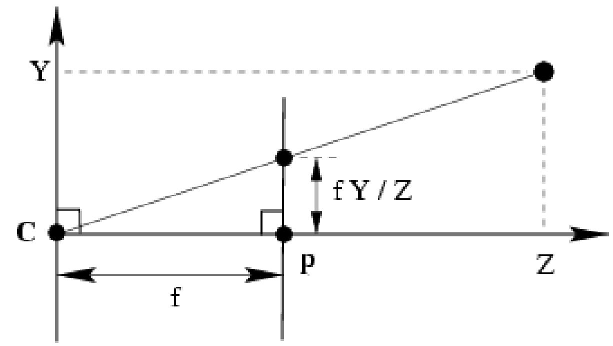


$$(X, Y, Z)^T \mapsto (fX/Z, fY/Z)^T$$

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Pinhole Camera Model



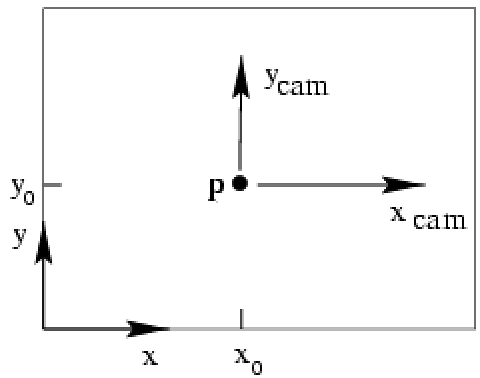


$$\begin{pmatrix} fX \\ fY \\ Z \end{pmatrix} = \begin{bmatrix} f & & \\ & f & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ & 1 & & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

$$\mathrm{x} = \mathrm{PX}$$

$$\mathrm{P} = \mathrm{diag}(f, f, 1)\left[\mathrm{I} \mid 0\right]$$

Principal Point Offset

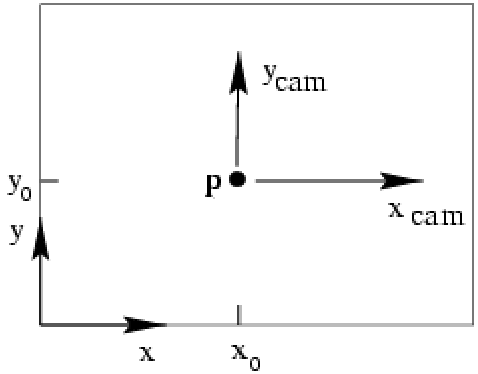


$$(X, Y, Z)^T \mapsto (fX/Z + p_x, fY/Z + p_y)^T$$

$(p_x, p_y)^T$ principal point

$$\begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} fX + Zp_x \\ fY + Zp_x \\ Z \end{pmatrix} = \begin{bmatrix} f & & p_x & 0 \\ & f & p_y & 0 \\ & & 1 & 0 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Principal Point Offset

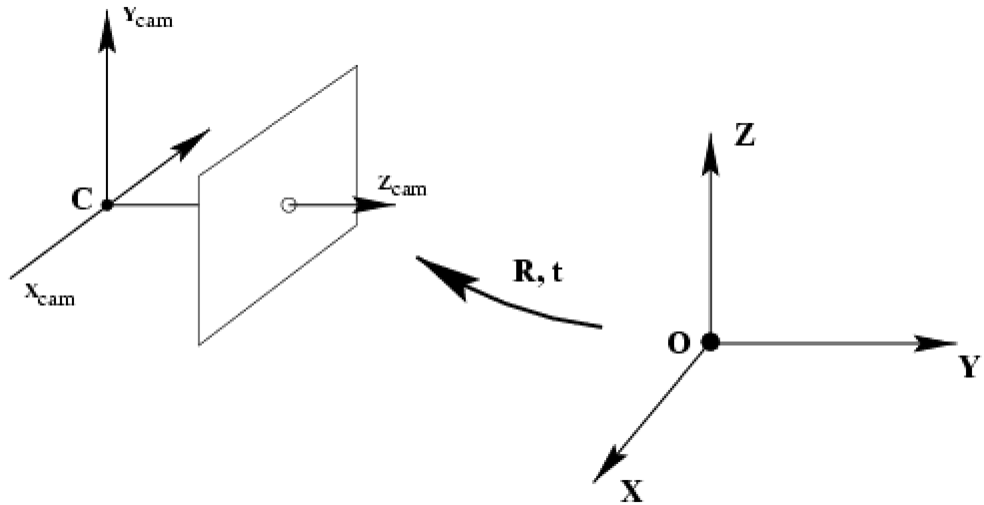


$$\mathrm{x} = \mathrm{K}\left[\mathrm{I} \mid 0\right]\mathrm{X}_{\mathrm{cam}}$$

$$K = \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$

Calibration Matrix

Camera Rotation and Translation

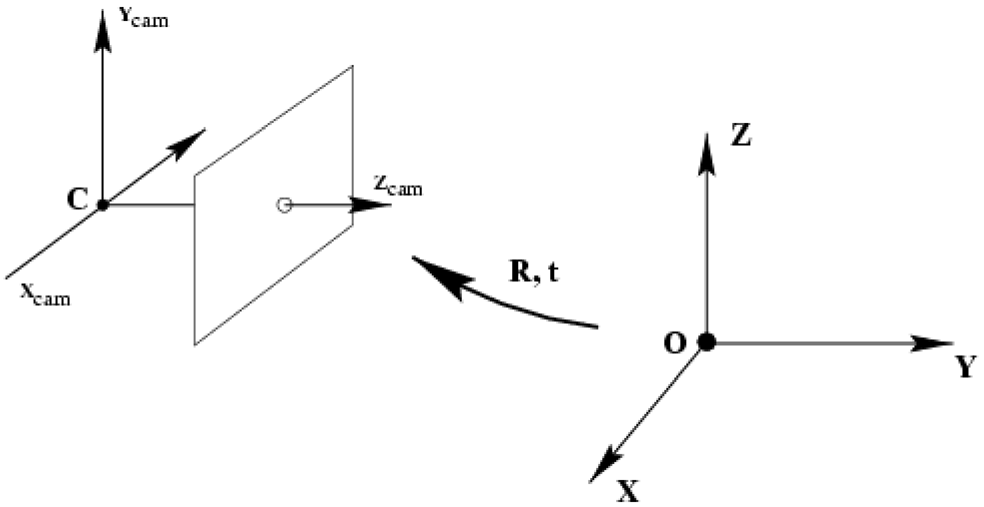


$$\tilde{X}_{cam} = R(\tilde{X} - \tilde{C})$$

$$X_{cam} = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix} = \begin{bmatrix} R & -RC \\ 0 & 1 \end{bmatrix} X$$

$$x = K[I \mid 0] X_{cam}$$

Camera Rotation and Translation



$$x = KR\left[I \mid -\tilde{C}\right]X$$

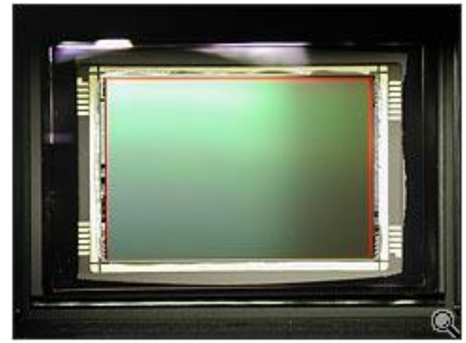
$$x = PX$$

$$P = K\left[R \mid t\right] \qquad t = -R\tilde{C}$$

Internal Camera Parameter
Internal Orientation
Intrinsic Matrix

External Camera Parameter
Exterior Orientation
Extrinsic Matrix

CCD Camera



Non-square pixels →

$$K = \begin{bmatrix} m_x & & \\ & m_y & \\ & & 1 \end{bmatrix} \begin{bmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} \alpha_x & & p_x \\ & \alpha_y & p_y \\ & & 1 \end{bmatrix}$$

Finite Projective Camera

$$K = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_x & p_y \\ & & 1 \end{bmatrix}$$

s: skew parameter,
=0 for most normal cameras

$$\mathrm{P} = \underbrace{\mathrm{KR}}_{}\left[\mathrm{I} \mid -\tilde{\mathrm{C}}\right]$$

11 dof (5+3+3)

non-singular

decompose P in K,R,C?

$$\mathrm{P} = \left[\mathrm{M} \mid \mathrm{p}_4\right] \qquad \left[\mathrm{K},\mathrm{R}\right] = RQ\left(\mathrm{M}\right) \quad \tilde{\mathrm{C}} = -\mathrm{M}^{-1}\mathrm{p}_4$$

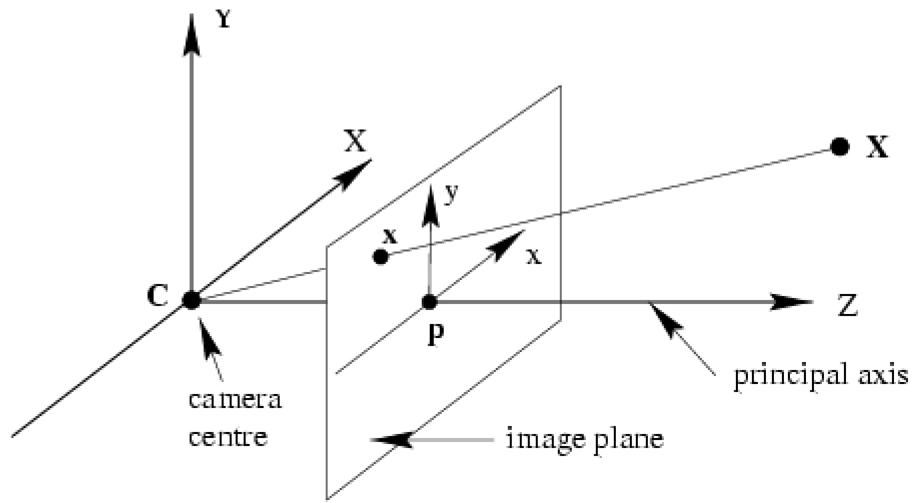
P_4: Last column of P

{finite cameras}={P_{4x3} | det M≠0}

If rank P=3, but rank M<3, then camera at infinity

Camera Anatomy

- Camera center
- Column points
- Principal plane
- Axis plane
- Principal point
- Principal ray



Camera Center

Camera Center (C): Null-space of camera projection matrix

$$\mathrm{PC} = 0$$

Proof:
$$\mathrm{X} = \lambda \mathrm{A} + (1-\lambda)\mathrm{C}$$

$$\mathrm{x} = \mathrm{PX} = \lambda \mathrm{PA} + (1-\lambda)\mathrm{PC}$$

For all A, all points on AC are projected on the same image point PA, therefore C is the camera center

Image of camera center is $(0,0,0)^T$, i.e. undefined

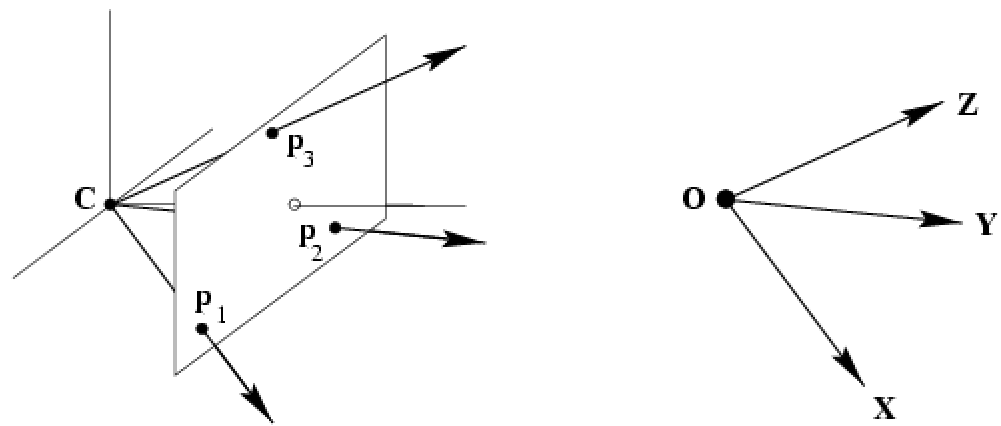
Finite cameras: $C = \begin{pmatrix} -\mathrm{M}^{-1}\mathrm{p}_4 \\ 1 \end{pmatrix}$

Infinite cameras: $C = \begin{pmatrix} \mathrm{d} \\ 0 \end{pmatrix}, \mathrm{Md} = 0$

The camera center is a point at infinity

Column Vectors

$$\left[p_2\right] = \left[p_1 p_2 p_3 p_4\right] \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$



p_1, p_2, p_3: vanishing points of the world coordinate X, Y, and Z axes
Image points corresponding to X,Y,Z directions and origin

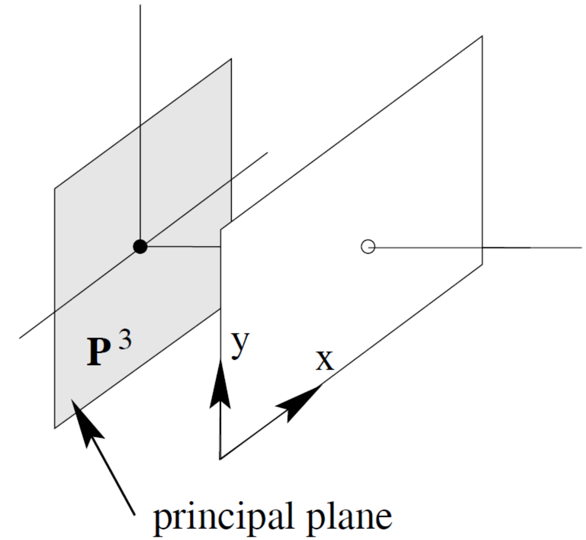
p_4 is the image of the world origin

Row Vectors

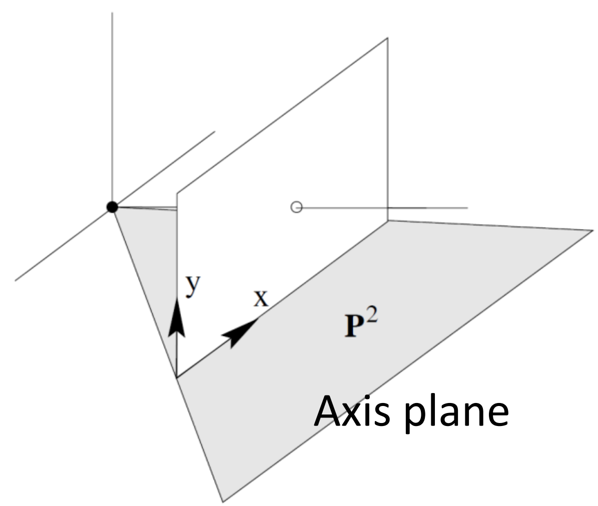
$$\mathtt{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{bmatrix} = \begin{bmatrix} \mathbf{P}^{1\mathsf{T}} \\ \mathbf{P}^{2\mathsf{T}} \\ \mathbf{P}^{3\mathsf{T}} \end{bmatrix}$$

Row Vectors

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} \mathrm{p}^{1\,\mathrm{T}} \\ \mathrm{p}^{2\,\mathrm{T}} \\ \mathrm{p}^{3\,\mathrm{T}} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

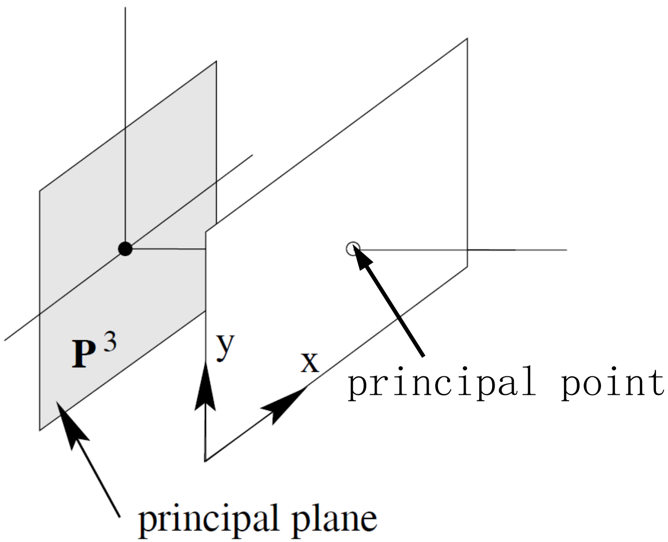


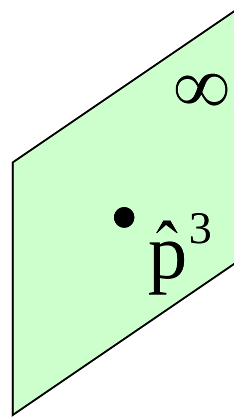
$$\begin{bmatrix} x \\ 0 \\ w \end{bmatrix} = \begin{bmatrix} \mathrm{p}^{1\,\mathrm{T}} \\ \mathrm{p}^{2\,\mathrm{T}} \\ \mathrm{p}^{3\,\mathrm{T}} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$



note: p^1,p^2 dependent on image reparametrization

The Principal Point



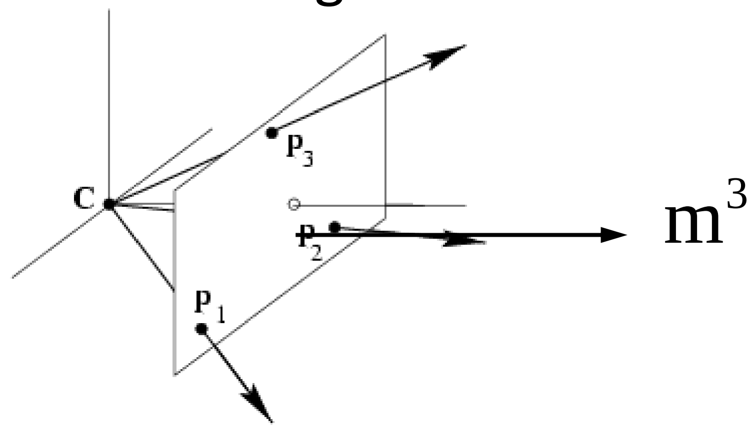


$$\hat{\mathrm{p}}^3 = \underbrace{\left(p_{31}, p_{32}, p_{33}\right.}_{m^3}\left.,0\right)$$

$$\mathrm{x}_0 = \mathrm{P}\hat{\mathrm{p}}^3 = \mathrm{M}\mathrm{m}^3$$

The Principal Axis Vector

vector defining front side of camera



$$x = P_{cam} X_{cam} = K[I \mid 0] X_{cam}$$

$$P_{cam} \mapsto k P_{cam}$$

$$v = \det(M) m^3 = (0,0,1)^T$$

$$v \mapsto k^4 v$$

(direction unaffected)

$$P = k K R[I \mid -\tilde{C}] = [M \mid p_4]$$

Direction unaffected because

$$\det(R) > 0$$

The principal axis vector ***v***=*det(M)m*^{3} is directed towards the front of the camera

Action of Projective Camera on Point

Forward projection

$$x = PX$$

$$x = PD = [M \mid p_4]D = Md$$

D: $(d^T, 0)^T$ point at the plane at infinity

Back-projection

$$PC = 0$$

$$X = P^+ x \qquad P^+ = P^T (PP^T)^{-1} \qquad PP^+ = I$$

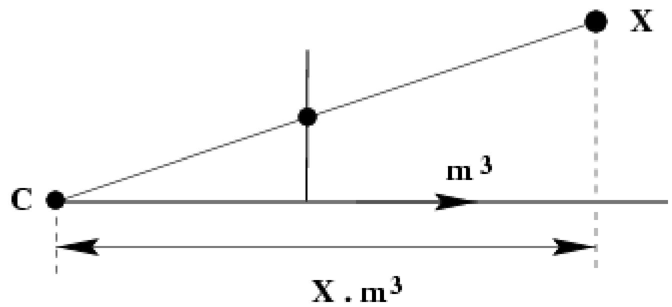
(pseudo-inverse)

$$X(\lambda) = P^+ x + \lambda C$$

For finite camera $d = M^{-1}x$

$$X(\lambda) = \mu \underbrace{\begin{pmatrix} M^{-1}x \\ 0 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} -M^{-1}p_4 \\ 1 \end{pmatrix}}_{C} = \begin{pmatrix} M^{-1}(\mu x - p_4) \\ 1 \end{pmatrix}$$

Depth of Points



$$w = \mathrm{P}^{3^{\mathrm{T}}}\mathrm{X} = \mathrm{P}^{3^{\mathrm{T}}}(\mathrm{X} - \mathrm{C}) = \mathrm{m}^{3^{\mathrm{T}}}(\tilde{\mathrm{X}} - \tilde{\mathrm{C}})$$

(PC=0) (dot product)

If $\det M > 0; \|\mathrm{m}^3\| = 1$

then m^3 unit vector in positive direction

$$\mathrm{depth}(\mathrm{X};\mathrm{P}) = \frac{\mathrm{sign}(\det\mathrm{M})\,w}{T\|\mathrm{m}^3\|}$$

$$\mathrm{X} = (X, Y, Z, T)^{\mathrm{T}}$$

Camera Matrix Decomposition

Finding the camera center

$$\mathrm{PC} = 0$$ (use SVD to find null-space)

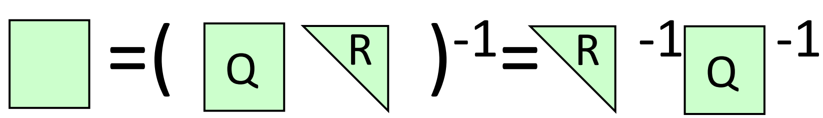
$$X = \det([\mathrm{p}_2, \mathrm{p}_3, \mathrm{p}_4]) \quad Y = -\det([\mathrm{p}_1, \mathrm{p}_3, \mathrm{p}_4])$$

$$Z = \det([\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_4]) \quad T = -\det([\mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3])$$

Finding the camera orientation and internal parameters

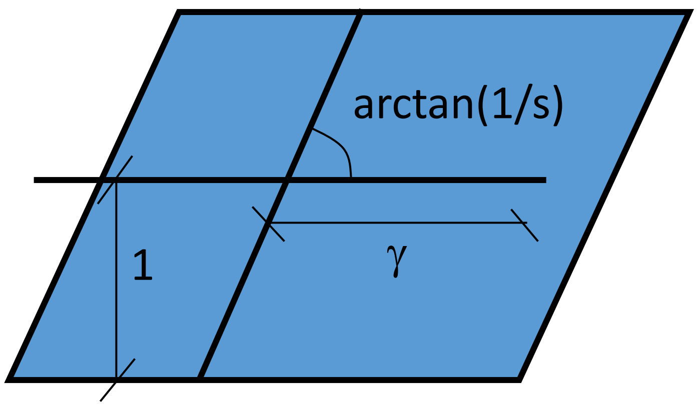
$$\mathrm{M} = \mathrm{KR}$$ (use RQ decomposition ~QR)

(if only QR, invert)



When is Skew Non-zero?

$$K = \begin{bmatrix} \alpha_x & s & p_x \\ & \alpha_x & p_y \\ & & 1 \end{bmatrix}$$



for CCD/CMOS, always s=0

Image from image, s≠0 possible
(non coinciding principal axis)

resulting camera: HP

Cameras at Infinity

Camera center at infinity

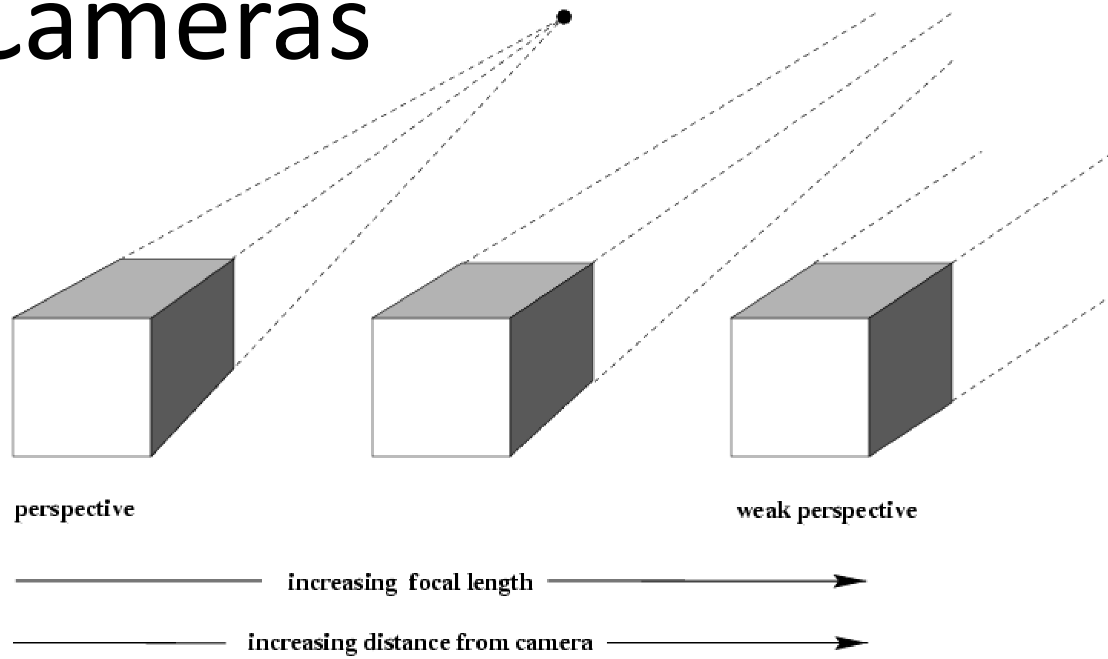
$$P\begin{bmatrix} \mathbf{d} \\ 0 \end{bmatrix} = 0 \Rightarrow \det M = 0$$

Affine and non-affine cameras

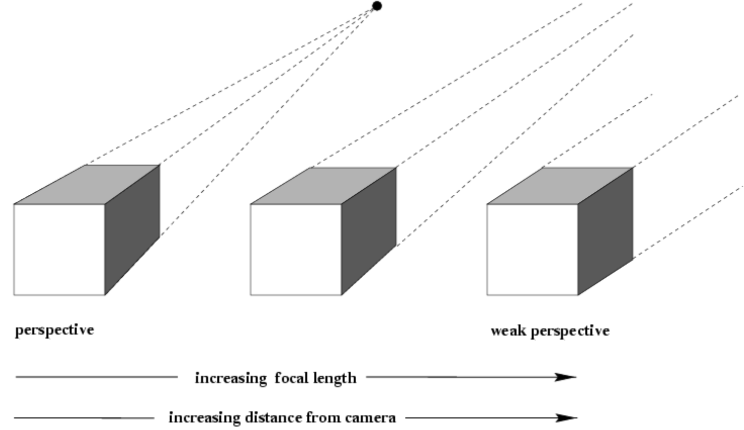
Definition: affine camera has $P^{3T}=(0,0,0,1)$

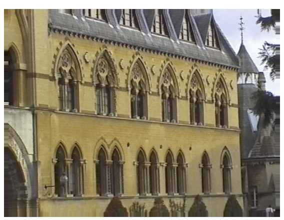
points at infinity are mapped to points at infinity

Affine Cameras



Affine Cameras





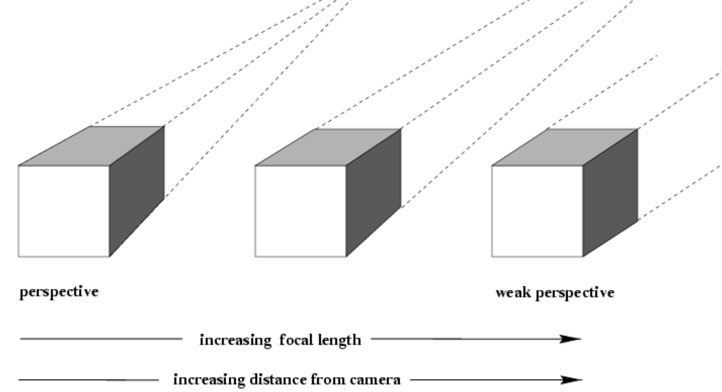
$$P_0 = KR\left[I \mid -\tilde{C}\right] = K\begin{bmatrix} r^{1T} & -r^{1T}\tilde{C} \\ r^{2T} & -r^{2T}\tilde{C} \\ r^{3T} & -r^{3T}\tilde{C} \end{bmatrix}$$

$$d_0 = -r^{3T}\tilde{C}$$

$$P_t = K\begin{bmatrix} r^{1T} & -r^{1T}\left(\tilde{C} - t r^3\right) \\ r^{2T} & -r^{2T}\left(\tilde{C} - t r^3\right) \\ r^{3T} & -r^{3T}\left(\tilde{C} - t r^3\right) \end{bmatrix} = K\begin{bmatrix} r^{1T} & -r^{1T}\tilde{C} \\ r^{2T} & -r^{2T}\tilde{C} \\ r^{3T} & d_t \end{bmatrix}$$

modifying p_{34} corresponds to moving along principal ray

Affine Cameras



now adjust zoom to compensate

$$P_t = K \begin{bmatrix} d_t/d_0 & & \\ & d_t/d_0 & \\ & & 1 \end{bmatrix} \begin{bmatrix} r^{1T} & -r^{1T}\tilde{C} \\ r^{2T} & -r^{2T}\tilde{C} \\ r^{3T} & d_t \end{bmatrix}$$

$$= \frac{d_t}{d_0} K \begin{bmatrix} r^{1T} & -r^{1T}\tilde{C} \\ r^{2T} & -r^{2T}\tilde{C} \\ r^{3T} d_0/d_t & d_0 \end{bmatrix}$$

$$P_\infty = \lim_{t\to\infty} P_t = K \begin{bmatrix} r^{1T} & -r^{1T}\tilde{C} \\ r^{2T} & -r^{2T}\tilde{C} \\ 0 & d_0 \end{bmatrix}$$

Summary Parallel Projection

$$P_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

canonical representation

$$K = \begin{bmatrix} K_{2\times 2} & 0 \\ 0 & 1 \end{bmatrix}$$

calibration matrix

principal point is not defined

A Hierarchy of Affine Cameras

$$P_\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$$

Orthographic projection

$$P_\infty = \begin{bmatrix} r^{1T} & t_1 \\ r^{1T} & t_2 \\ 0 & 1 \end{bmatrix} \qquad \text{(5dof)}$$

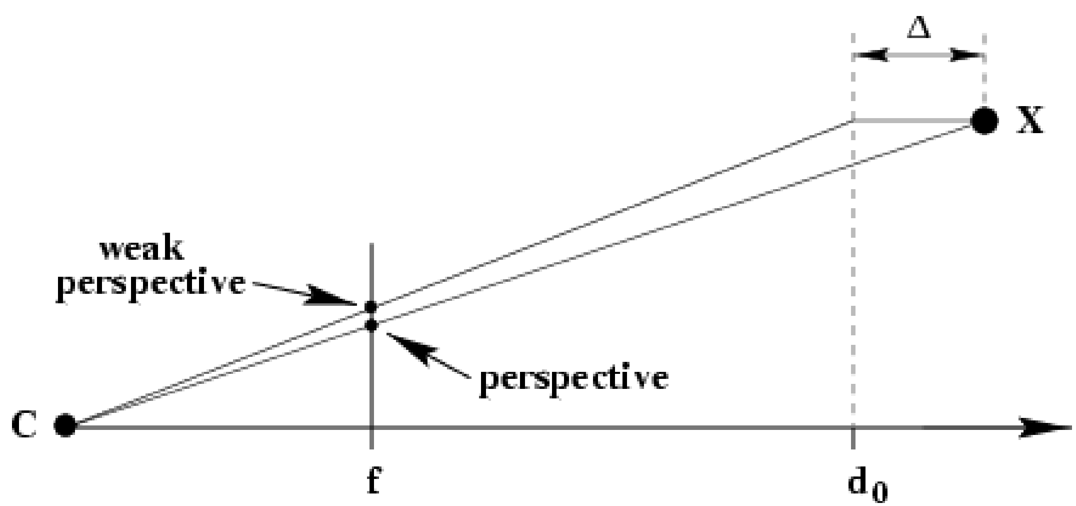
Scaled orthographic projection

$$P_\infty = \begin{bmatrix} r^{1T} & t_1 \\ r^{1T} & t_2 \\ 0 & 1/k \end{bmatrix} \qquad \text{(6dof)}$$

A Hierarchy of Affine Cameras

Weak perspective projection

$$P_\infty = \begin{bmatrix} \alpha_x & & \\ & \alpha_y & \\ & & 1 \end{bmatrix} \begin{bmatrix} \mathrm{r}^{1\mathrm{T}} & t_1 \\ \mathrm{r}^{1\mathrm{T}} & t_2 \\ 0 & 1/k \end{bmatrix} \quad \text{(7dof)}$$



A Hierarchy of Affine Cameras

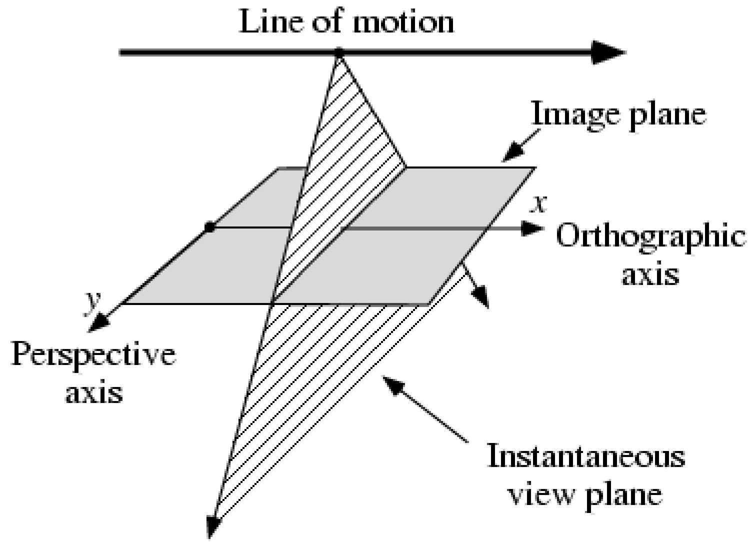
Affine camera (8dof)

$$P_A = \begin{bmatrix} \alpha_x & s & \\ & \alpha_y & \\ & & 1 \end{bmatrix} \begin{bmatrix} r^{1T} & t_1 \\ r^{1T} & t_2 \\ 0 & 1/k \end{bmatrix} \quad P_A = \begin{bmatrix} m_{11} & m_{12} & m_{13} & t_1 \\ m_{21} & m_{22} & m_{23} & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P_A = \left[3\times 3\,\text{affine}\right] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left[4\times 4\,\text{affine}\right]$$

1. Affine camera=camera with principal plane coinciding with Π_∞
2. Affine camera maps parallel lines to parallel lines
3. No center of projection, but direction of projection $P_A D=0$ (point on Π_∞)

Pushbroom Cameras



(11dof)

$$\mathrm{X} = (X, Y, X, T)^{\mathrm{T}} \qquad \mathrm{PX} = (x, y, w)^{\mathrm{T}} \qquad (x, y / w)^{\mathrm{T}}$$

$$\tilde{x} = x = \mathrm{P}^{1\mathsf{T}}\mathbf{X} \qquad \tilde{y} = y/z = \frac{\mathrm{P}^{2\mathsf{T}}\mathbf{X}}{\mathrm{P}^{3\mathsf{T}}\mathbf{X}}$$

Straight lines are not mapped to straight lines!
(otherwise it would be a projective camera)