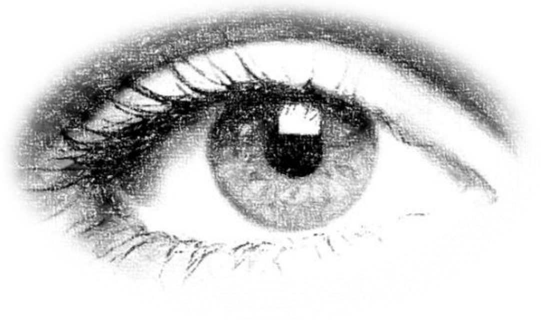

Projective Geometry

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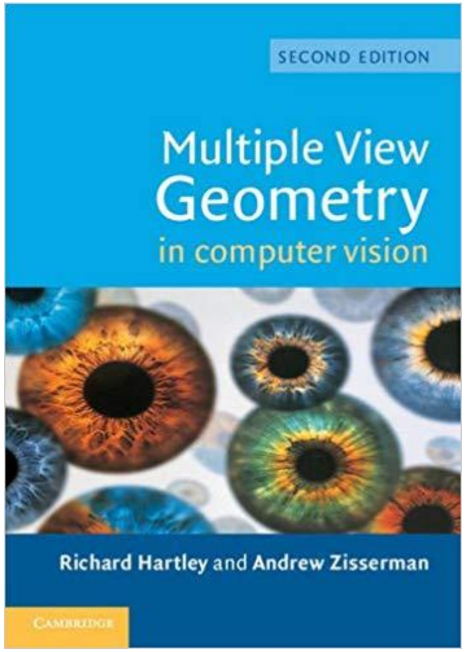
Outline

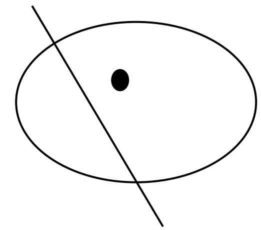
- Projective 2D geometry
- Projective 3D geometry



[Slides credit: Marc Pollefeys]

Projective 2D Geometry

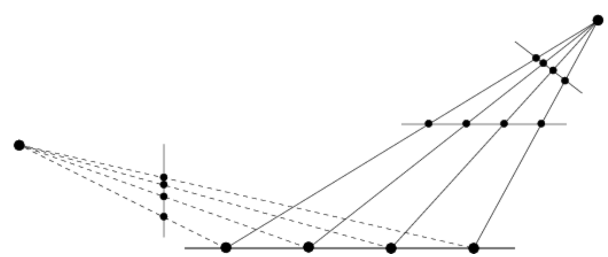
- Points, lines & conics



- Transformations & invariants

- 1D projective geometry and the cross-ratio



Homogeneous Coordinates

- Homogeneous representation of lines

$$ax + by + c = 0 \qquad (a,b,c)^{\mathsf{T}}$$

$$(ka)x + (kb)y + kc = 0, \forall k \neq 0 \qquad (a,b,c)^{\mathsf{T}} \sim k(a,b,c)^{\mathsf{T}}$$

equivalence class of vectors, any vector is representative

- Homogeneous representation of points

$\mathrm{x} = (x, y)^{\mathsf{T}}$ on $\mathrm{l} = (a,b,c)^{\mathsf{T}}$ if and only if $ax + by + c = 0$

$$(x,y,1)(a,b,c)^{\mathsf{T}} = (x,y,1)\mathrm{l} = 0 \qquad (x, y, 1)^{\mathsf{T}} \sim k(x, y, 1)^{\mathsf{T}}, \forall k \neq 0$$

The point x lies on the line l if and only if $\mathrm{x}^{\mathsf{T}}\mathrm{l}=\mathrm{l}^{\mathsf{T}}\mathrm{x}=0$

Homogeneous coordinates $(x_1, x_2, x_3)^{\mathsf{T}}$ but only 2DOF

Inhomogeneous coordinates $(x, y)^{\mathsf{T}}$

The point x=$(x_1, x_2, x_3)^{\mathrm{T}}$ represent the point $(x_1/x_3, x_2/x_3)^{\mathrm{T}}$ in $\mathbb{R}^2$

Points and Lines

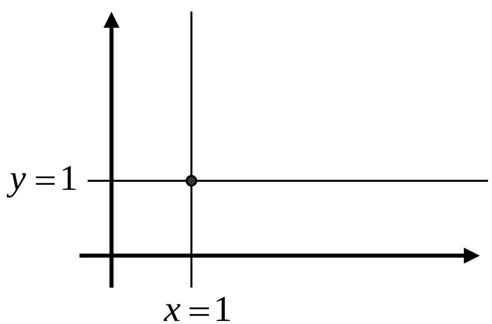
- Intersections of lines

  The intersection of two lines $\mathbf{l}$ and $\mathbf{l}'$ is $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$

- Line joining two points

  The line through two points $\mathbf{x}$ and $\mathbf{x}'$ is $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$

Example

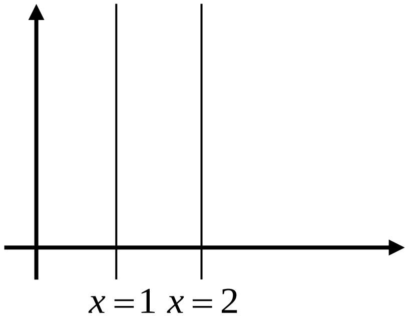


Ideal Points and the Line at Infinity

- Intersections of parallel lines

$$\mathrm{l} = (a, b, c)^\mathsf{T} \text{ and } \mathrm{l}' = (a, b, c')^\mathsf{T} \qquad \mathrm{l} \times \mathrm{l}' = (b, -a, 0)^\mathsf{T}$$

Example



$(b, -a)$ tangent vector (line’s direction)

(a, b) normal direction

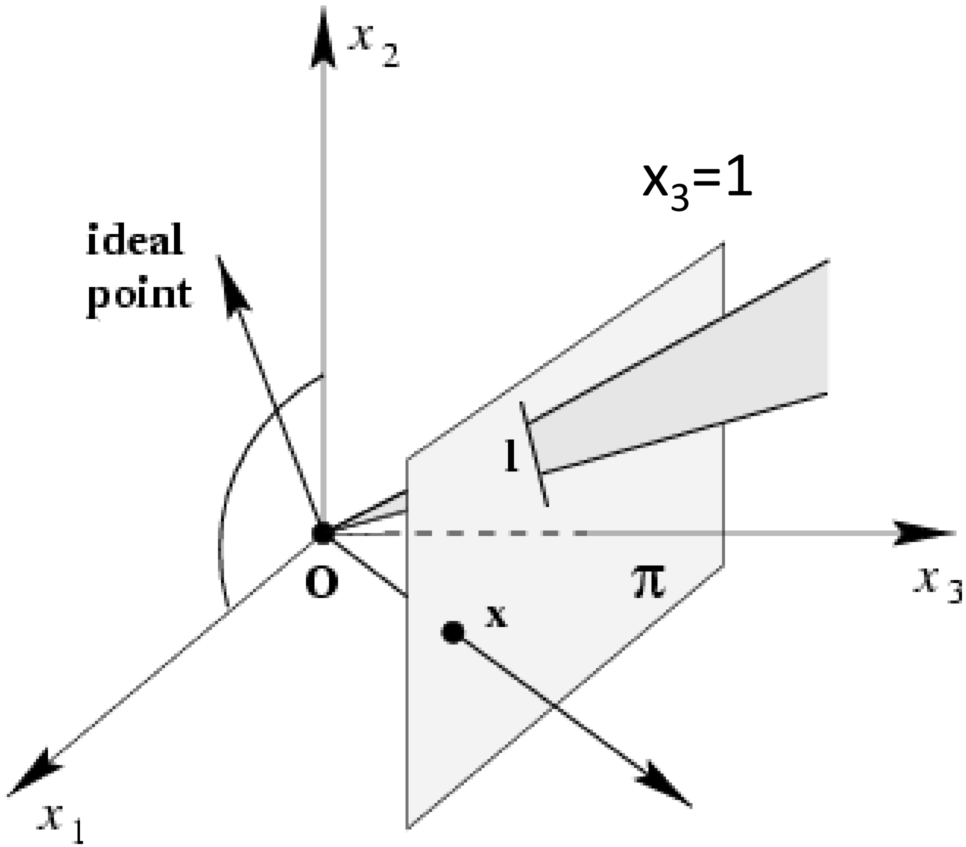
Ideal points $(x_1, x_2, 0)^\mathsf{T}$

Line at infinity $\mathrm{l}_\infty = (0, 0, 1)^\mathsf{T}$

$$\mathbf{P}^2 = \mathbf{R}^2 \cup \mathrm{l}_\infty$$

Note that in **P**^{2} there is no distinction between ideal points and others

A Model for the Projective Plane



exactly one line through two points

exaclty one point at intersection of two lines

Duality

$$\mathbf{x} \longleftrightarrow \mathbf{l}$$

$$\mathbf{x}^{\mathsf{T}}\mathbf{l} = 0 \longleftrightarrow \mathbf{l}^{\mathsf{T}}\mathbf{x} = 0$$

$$\mathbf{x} = \mathbf{l} \times \mathbf{l}' \longleftrightarrow \mathbf{l} = \mathbf{x} \times \mathbf{x}'$$

- Duality principle:

  To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem

Conics

Curve described by 2nd-degree equation in the plane

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

or *homogenized* $x \mapsto x_1/x_3,\ y \mapsto x_2/x_3$

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

or in matrix form

$$\mathrm{x}^{\mathsf{T}}\mathbf{C}\,\mathrm{x} = 0 \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

symmetric

5DOF: $\{a : b : c : d : e : f\}$

Five Points Define a Conic

For each point the conic passes through

$$a x_i^2 + b x_i y_i + c y_i^2 + d x_i + e y_i + f = 0$$

or

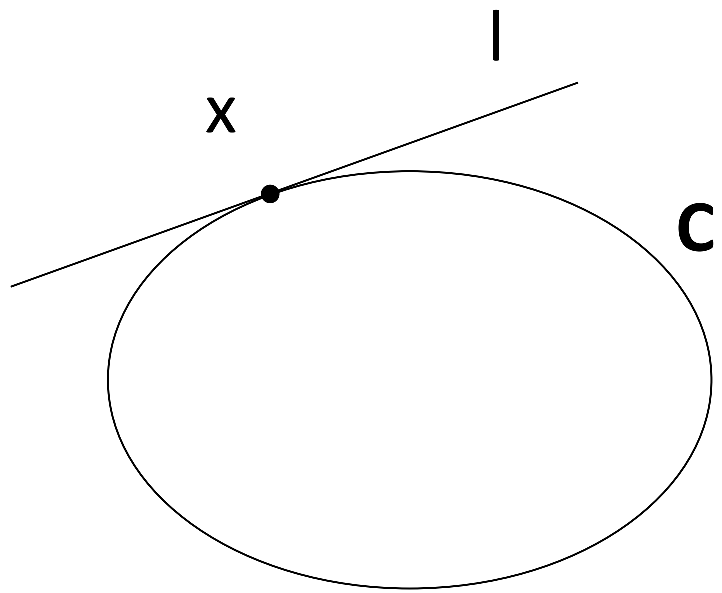
$$\left(x_i^2, x_i y_i, y_i^2, x_i, y_i, f\right)\mathbf{c} = 0 \qquad \mathbf{c} = \left(a, b, c, d, e, f\right)^{\mathsf{T}}$$

stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4 y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5 y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix}\mathbf{c} = 0$$

Tangent Lines to Conics

The line $\mathbf{l}$ tangent to $\mathbf{C}$ at point $\mathbf{x}$ on $\mathbf{C}$ is given by $\mathbf{l}=\mathbf{C}\mathbf{x}$

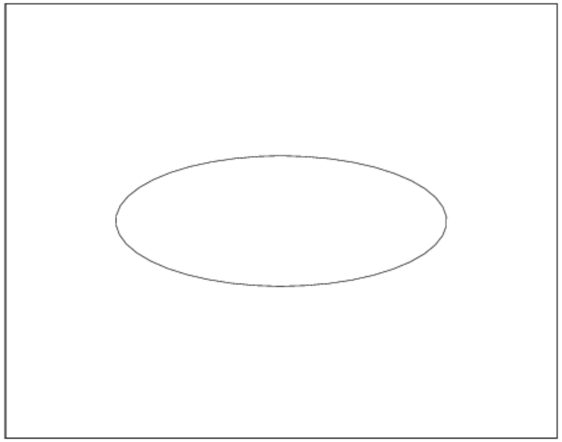


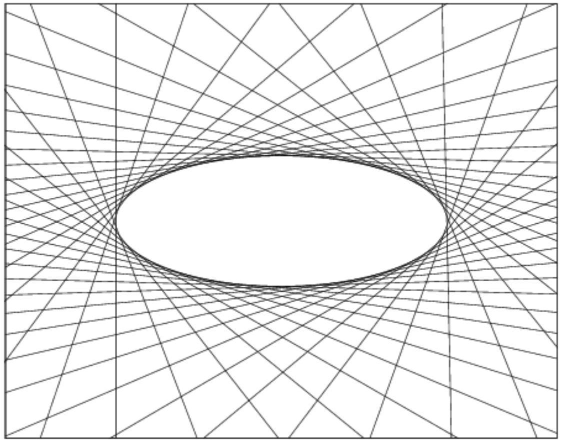
Dual Conics

A line tangent to the conic **C** satisfies $\mathbf{l}^{\mathsf{T}}\,\mathbf{C}^{*}\,\mathbf{l} = 0$

In general (**C** full rank): $\mathbf{C}^{*} = \mathbf{C}^{-1}$

Dual conics = line conics = conic envelopes





Projective Transformations

Definition:

A *projectivity* is an invertible mapping h from P^2 to itself such that three points x_1,x_2,x_3 lie on the same line if and only if $h(x_1),h(x_2),h(x_3)$ do.

Theorem:

A mapping $h:P^2 \rightarrow P^2$ is a projectivity if and only if there exist a non-singular 3x3 matrix **H** such that for any point in P^2 reprented by a vector x it is true that $h(x)=\mathbf{H}x$

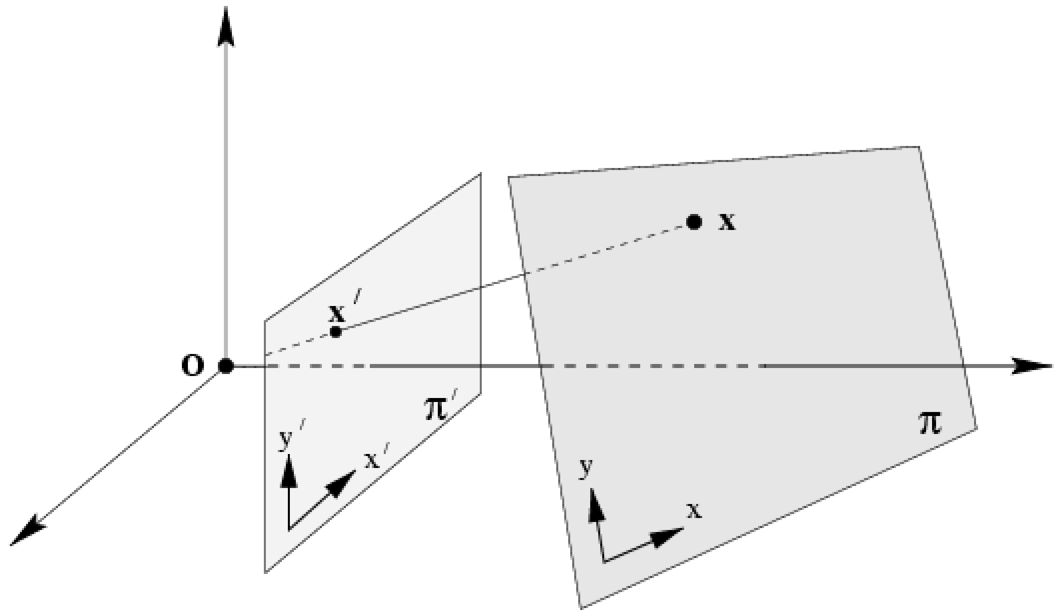
Definition: Projective transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad \mathbf{x}' = \mathbf{H}\,\mathbf{x}$$

8DOF

projectivity=collineation=projective transformation=homography

Mapping between Planes



central projection may be expressed by x’=Hx
(application of theorem)

Removing Projective Distortion

select four points in a plane with know coordinates

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

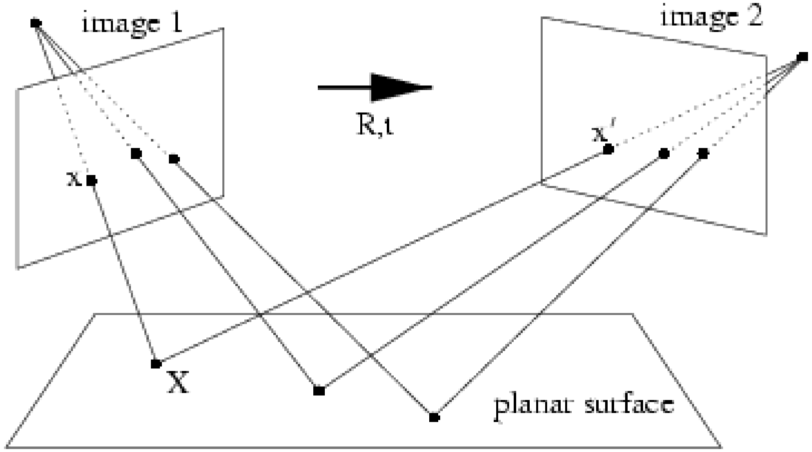
$$x'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{11}x + h_{12}y + h_{13}$$
$$y'\left(h_{31}x + h_{32}y + h_{33}\right) = h_{21}x + h_{22}y + h_{23}$$

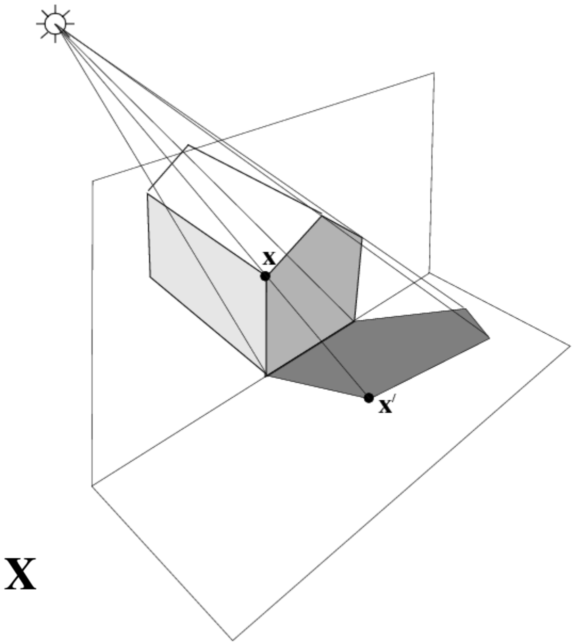
(linear in h_{ij})

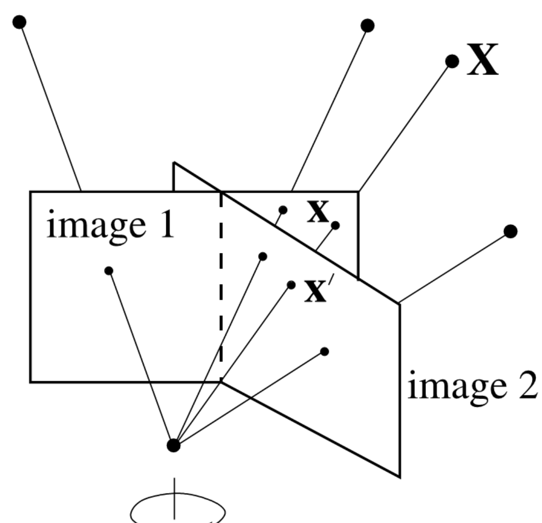
(2 constraints/point, 8DOF $\Rightarrow$ 4 points needed)

Remark: no calibration at all necessary

More Examples







Transformation of Lines and Conics

For a point transformation

$$\mathrm{x}' = \mathbf{H}\,\mathrm{x}$$

Transformation for lines

$$\mathrm{l}' = \mathbf{H}^{-\mathsf{T}}\,\mathrm{l}$$

Transformation for conics

$$\mathbf{C}' = \mathbf{H}^{-\mathsf{T}}\mathbf{C}\mathbf{H}^{-1}$$

Transformation for dual conics

$$\mathbf{C}'^{*} = \mathbf{H}\mathbf{C}^{*}\mathbf{H}^{\mathsf{T}}$$

A Hierarchy of Transformations

- Projective linear group
- Affine group (last row (0,0,1))
- Euclidean group (upper left 2x2 orthogonal)
- Oriented Euclidean group (upper left 2x2 det 1)

Alternative, characterize transformation in terms of elements or quantities that are preserved or *invariant*

e.g. Euclidean transformations leave distances unchanged

Similarity

Affine

Projective

Class I: Isometries

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \varepsilon\cos\theta & -\sin\theta & t_x \\ \varepsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \varepsilon = \pm 1$$

orientation preserving: $\varepsilon = 1$ **(Euclidean transform)**

orientation reversing: $\varepsilon = -1$

$$\mathrm{x}' = \mathbf{H}_E\,\mathrm{x} = \begin{bmatrix} \mathbf{R} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix} \mathrm{x} \qquad \mathbf{R}^{\mathsf{T}}\mathbf{R} = \mathbf{I}$$

3DOF (1 rotation, 2 translation), can be computed from 2 point correspondences

special cases: pure rotation, pure translation

Invariants: length, angle, area

Class II: Similarities

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\mathrm{x}' = \mathbf{H}_S\ \mathrm{x} = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \mathrm{x} \qquad \mathbf{R}^\mathsf{T}\mathbf{R} = \mathbf{I}$$

4DOF (1 scale, 1 rotation, 2 translation),
can be computed from 2 point correspondences

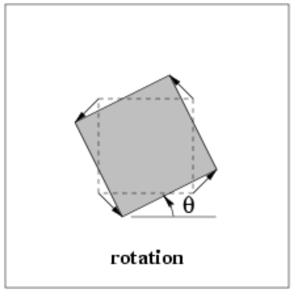
also know as *equi-form* (shape preserving)

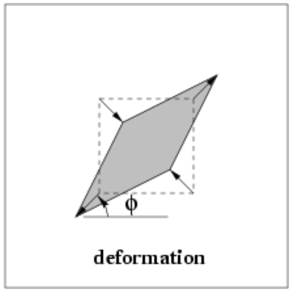
Invariants: ratios of length, angle, ratios of areas, parallel lines

Class III: Affine Transformations

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\mathrm{x}' = \mathbf{H}_A \, \mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \mathrm{x}$$





$$\mathbf{A} = \mathbf{R}(\theta)\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi) \qquad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

6DOF (2 scale, 2 rotation, 2 translation),

can be computed from 3 point correspondences

non-isotropic scaling! (2DOF: scale ratio and orientation)

Invariants: parallel lines, ratios of parallel lengths, ratios of areas

Class VI: Projective Transformations

$$\mathrm{x}' = \mathbf{H}_P\,\mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix} \mathrm{x} \qquad \mathrm{v} = \left(v_1, v_2\right)^\mathsf{T}$$

8DOF (2 scale, 2 rotation, 2 translation, 2 line at infinity)

can be computed from 4 point correspondences

Action non-homogeneous over the plane

Invariants: cross-ratio of four points on a line, (ratio of ratio)

Action of Affinities and Projectivities on Line at Infinity

$$\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^\mathsf{T} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

Line at infinity stays at infinity, but points move along line

$$\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$

Line at infinity becomes finite, allows to observe vanishing points, horizon

Decomposition of Projective Transformations

$$\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix}$$

S: similarity

A: Affine

P: Projective

$$\mathbf{A} = s\mathbf{R}\mathbf{K} + \mathrm{t}\mathrm{v}^\mathsf{T}$$

decomposition unique (if chosen s>0)

$\mathbf{K}$ upper-triangular, $\det \mathbf{K} = 1$

Example:

$$\mathbf{H} = \begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

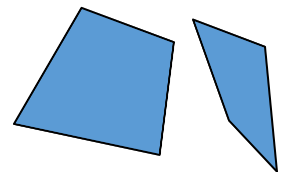
$$\mathbf{H} = \begin{bmatrix} 2\cos 45^\circ & -2\sin 45^\circ & 1.0 \\ 2\sin 45^\circ & 2\cos 45^\circ & 2.0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Summary of Transformations

Invariant Properties

Projective
8dof

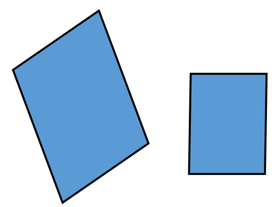
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio

Affine
6dof

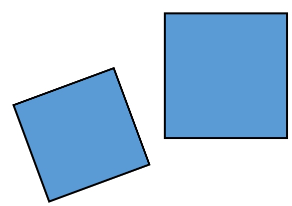
$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Parallellism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids).
The line at infinity l_∞

Similarity
4dof

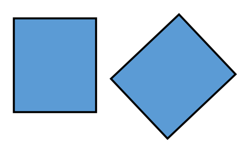
$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Ratios of lengths, angles.
The circular points I,J

Euclidean
3dof

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

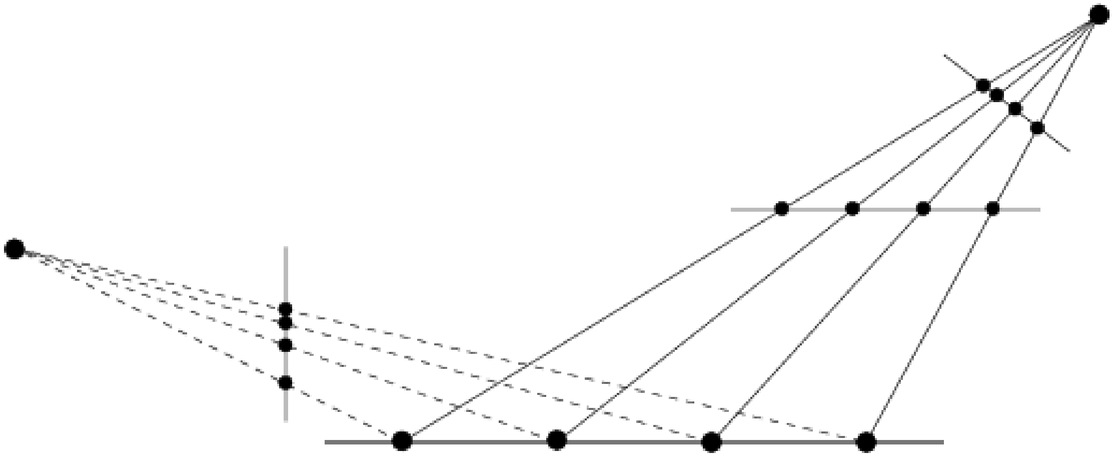


lengths, areas.

Cross Ratio

$$Cross(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) = \frac{|\bar{x}_1, \bar{x}_2||\bar{x}_3, \bar{x}_4|}{|\bar{x}_1, \bar{x}_3||\bar{x}_2, \bar{x}_4|} \qquad |\bar{x}_i, \bar{x}_j| = \det\begin{bmatrix} x_{i1} & x_{j1} \\ x_{i2} & x_{j2} \end{bmatrix}$$

Invariant under projective transformations



Number of Invariants?

The number of functional invariants is equal to, or greater than, the number of degrees of freedom of the configuration less the number of degrees of freedom of the transformation

e.g. configuration of 4 points in general position has 8 dof (2/pt) and so 4 similarity, 2 affinity and zero projective invariants since these transformations have respectively 4, 6 and 8 degrees of freedom

Recovering Metric and Affine Properties from Images

- Parallelism
- Parallel length ratios

- Angles
- Length ratios

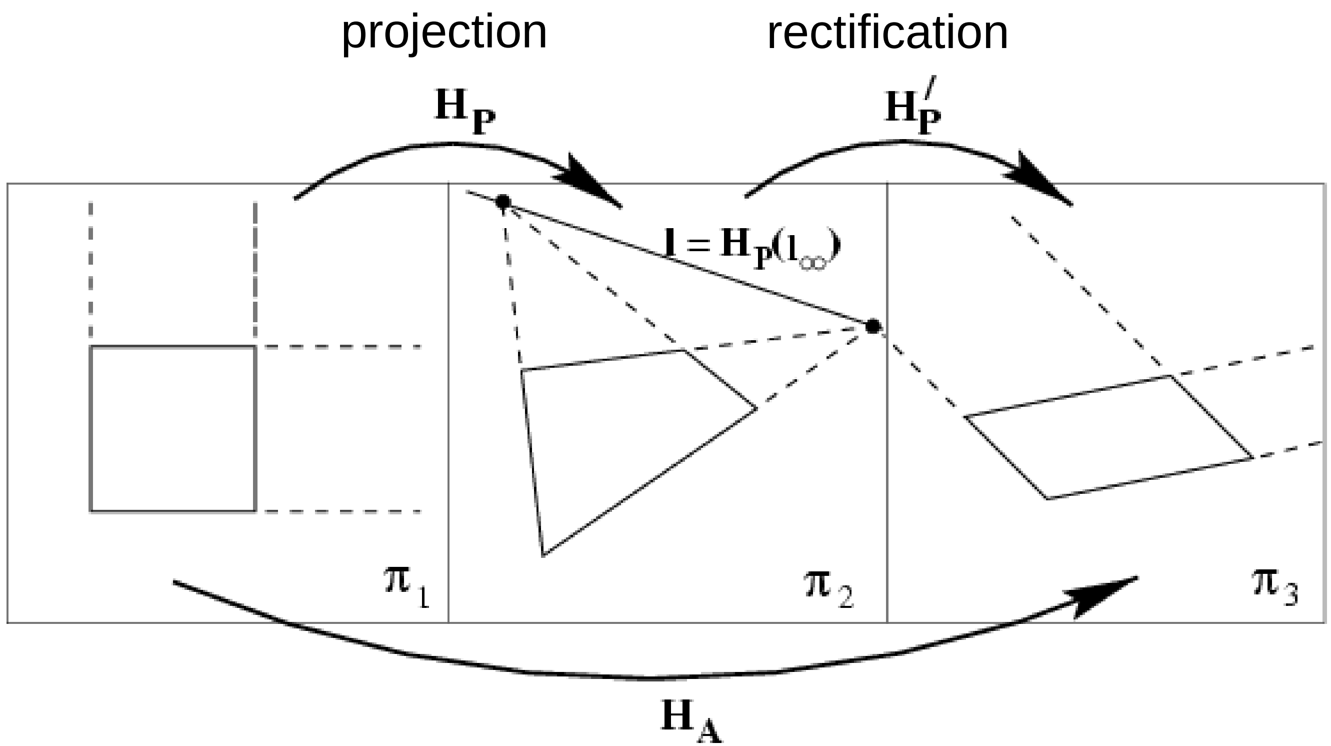
The Line at Infinity

$$\mathbf{l}'_\infty = \mathtt{H}_\mathtt{A}^{-\mathsf{T}}\mathbf{l}_\infty = \begin{bmatrix} \mathtt{A}^{-\mathsf{T}} & \mathbf{0} \\ -\mathbf{t}^\mathsf{T}\mathtt{A}^{-\mathsf{T}} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{l}_\infty$$

The line at infinity l_∞ is a fixed line under a projective transformation H if and only if H is an affinity

Note: not fixed pointwise

Affine Properties from Images

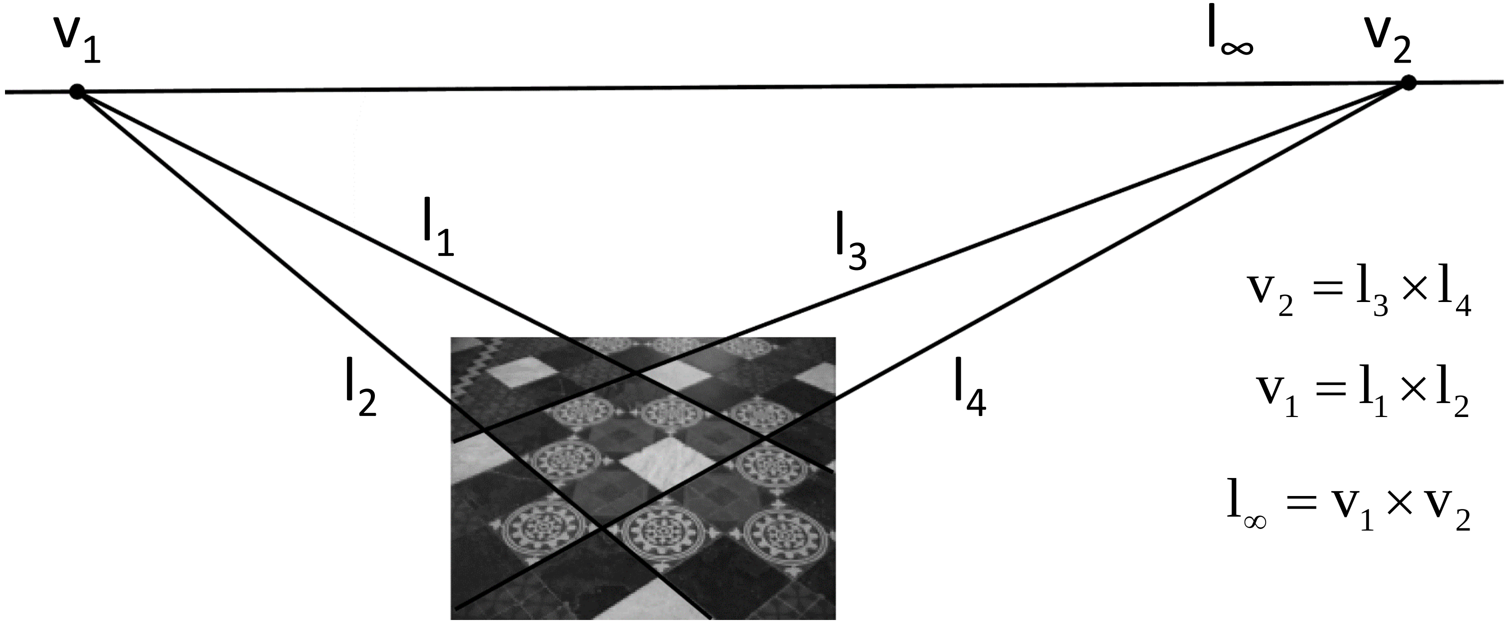


$$\mathbf{H}_{PA} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}}_{\mathrm{H_P}} \mathbf{H}_A$$

$$\mathrm{l}_\infty = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^{\mathrm{T}}, l_3 \neq 0$$

$$H_P^{-\mathrm{T}}(l_1, l_2, l_3)^{\mathrm{T}} = (0,0,1)^{\mathrm{T}} = l_\infty$$

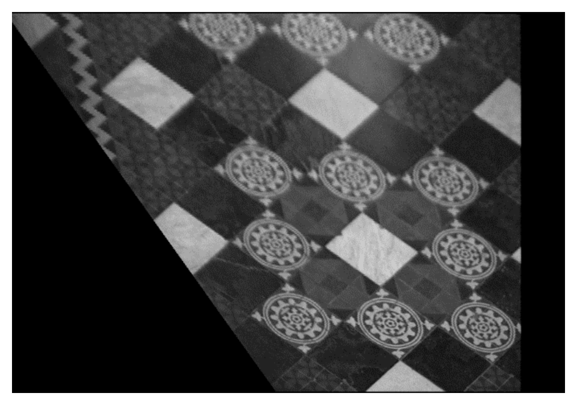
Affine Rectification



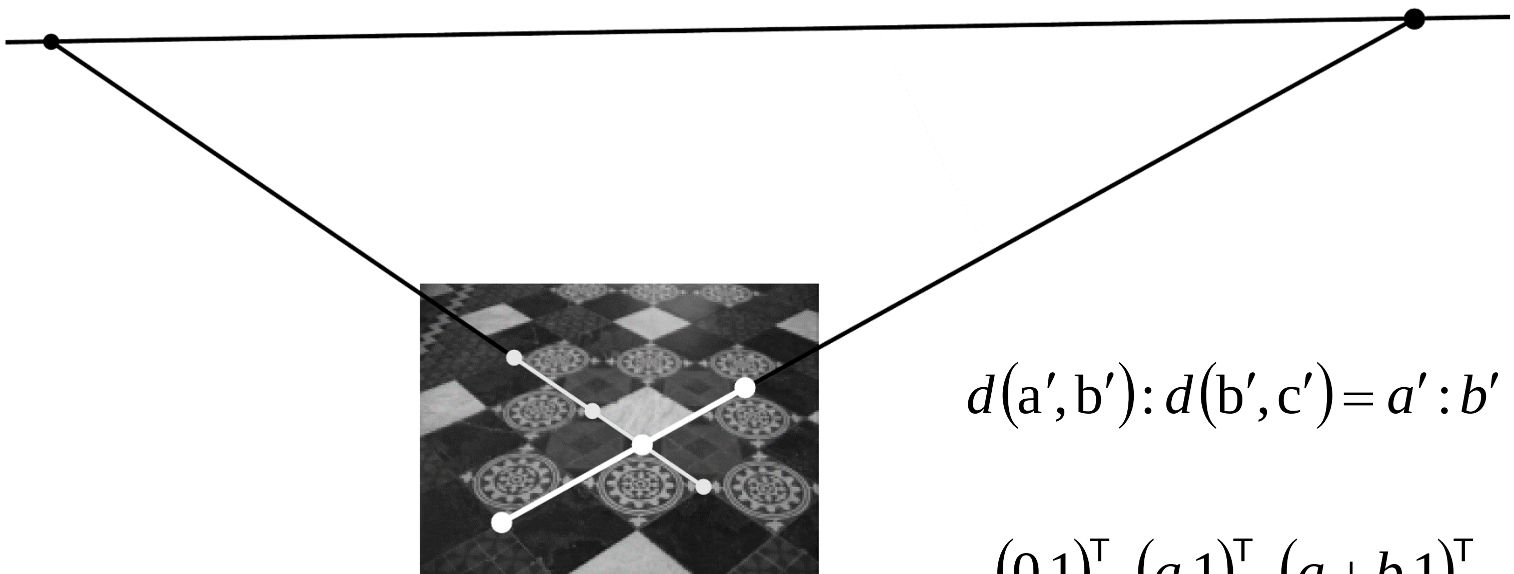
$$v_2 = l_3 \times l_4$$

$$v_1 = l_1 \times l_2$$

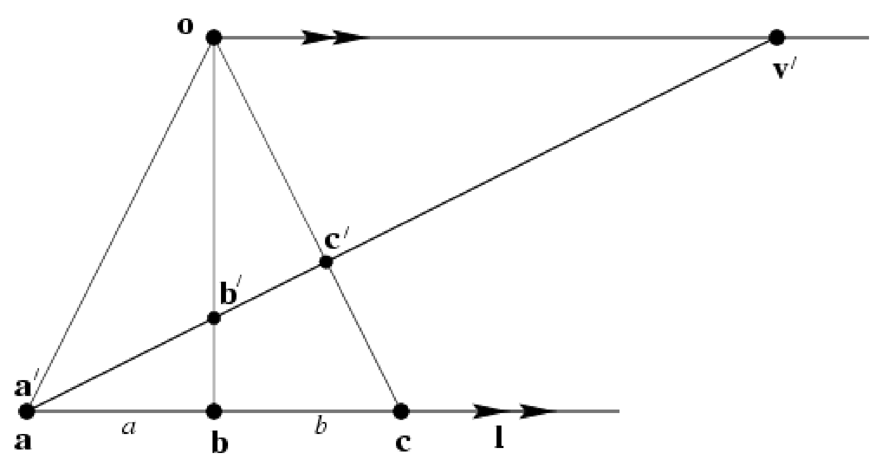
$$l_\infty = v_1 \times v_2$$



Distance Ratios



$$d(\mathrm{a}', \mathrm{b}') : d(\mathrm{b}', \mathrm{c}') = a' : b'$$



$$(0,1)^{\mathsf{T}}, (a,1)^{\mathsf{T}}, (a+b,1)^{\mathsf{T}}$$

$$\downarrow \mathbf{H}$$

$$\mathrm{a}', \mathrm{b}', \mathrm{c}'$$

$$\mathrm{v}' = \mathbf{H}(1,0)^{\mathsf{T}}$$

The Circular Points

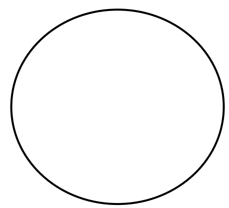
$$\mathrm{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \quad \mathrm{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

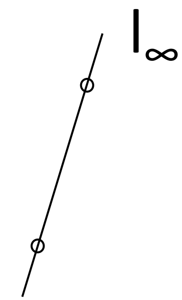
$$\mathrm{I}' = \mathbf{H}_S\,\mathrm{I} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = se^{i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathrm{I}$$

The circular points I, J are fixed points under the projective transformation **H** iff **H** is a similarity

The Circular Points

"circular points"





$$x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

$$x_3 = 0$$

$$x_1^2 + x_2^2 = 0$$

$$\mathrm{I} = (1, i, 0)^\mathsf{T}$$

$$\mathrm{J} = (1, -i, 0)^\mathsf{T}$$

Algebraically, encodes orthogonal directions

$$\mathrm{I} = (1,0,0)^\mathsf{T} + i(0,1,0)^\mathsf{T}$$

Conic Dual to the Circular Points

$$\mathbf{C}_\infty^* = \mathrm{IJ}^\mathsf{T} + \mathrm{JI}^\mathsf{T} \qquad \mathbf{C}_\infty^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{C}_\infty^* = \mathbf{H}_S \mathbf{C}_\infty^* \mathbf{H}_S^\mathsf{T}$$

The dual conic $\mathbf{C}_\infty^*$ is fixed conic under the projective transformation **H** iff **H** is a similarity

Angles

Euclidean: $\mathrm{l} = (l_1, l_2, l_3)^{\mathsf{T}}$ $\quad$ $\mathrm{m} = (m_1, m_2, m_3)^{\mathsf{T}}$

$$\cos\theta = \frac{l_1 m_1 + l_2 m_2}{\sqrt{\left(l_1^2 + l_2^2\right)\left(m_1^2 + m_2^2\right)}}$$

Projective: $$\cos\theta = \frac{\mathrm{l}^{\mathsf{T}}\, \mathbf{C}_\infty^*\, \mathrm{m}}{\sqrt{\left(\mathrm{l}^{\mathsf{T}}\, \mathbf{C}_\infty^*\, \mathrm{l}\right)\left(\mathrm{m}^{\mathsf{T}}\, \mathbf{C}_\infty^*\, \mathrm{m}\right)}}$$

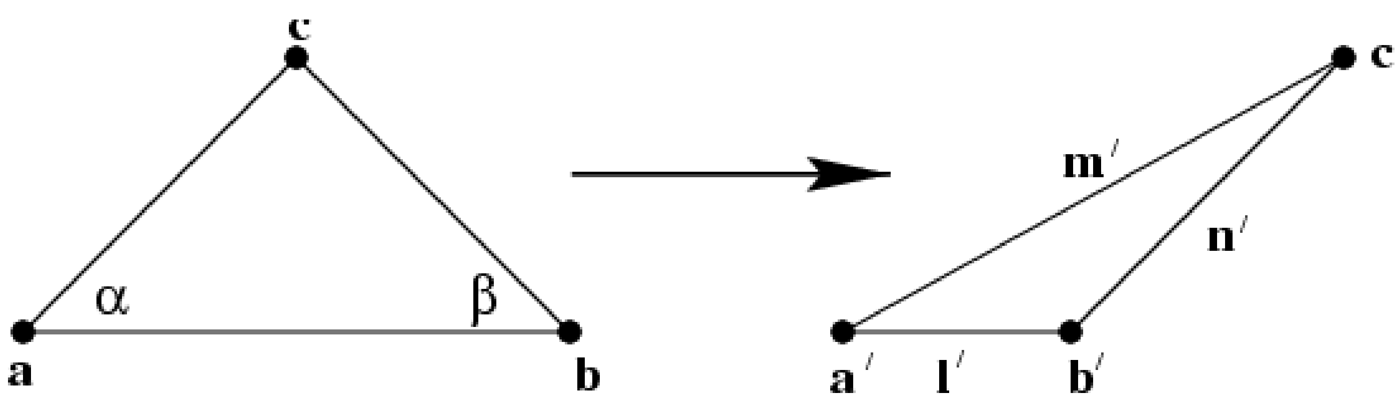
(This equation is Invariant to projective transform)

$\mathrm{l}^{\mathsf{T}}\, \mathbf{C}_\infty^*\, \mathrm{m} = 0$ If orthogonal

Length Ratios

$$\frac{d(b,c)}{d(a,c)} = \frac{\sin\alpha}{\sin\beta}$$

$\cos\alpha$ and $\cos\beta$ can be derived with the equations in the previous page



Metric Properties from Images

$$\mathbf{C}_\infty^*{}' = \left(\mathbf{H}_P\mathbf{H}_A\mathbf{H}_S\right)\mathbf{C}_\infty^*\left(\mathbf{H}_P\mathbf{H}_A\mathbf{H}_S\right)^\mathsf{T}$$
$$= \left(\mathbf{H}_P\mathbf{H}_A\right)\mathbf{H}_S\mathbf{C}_\infty^*\mathbf{H}_S^\mathsf{T}\left(\mathbf{H}_P\mathbf{H}_A\right)^\mathsf{T}$$
$$= \left(\mathbf{H}_P\mathbf{H}_A\right)\mathbf{C}_\infty^*\left(\mathbf{H}_P\mathbf{H}_A\right)^\mathsf{T}$$
$$= \begin{bmatrix} \mathbf{K}\mathbf{K}^\mathsf{T} & \mathbf{K}^\mathsf{T}\mathrm{v} \\ \mathrm{v}^\mathsf{T}\mathbf{K} & \mathrm{v}^\mathsf{T}\mathrm{v} \end{bmatrix}$$

Rectifying transformation from SVD

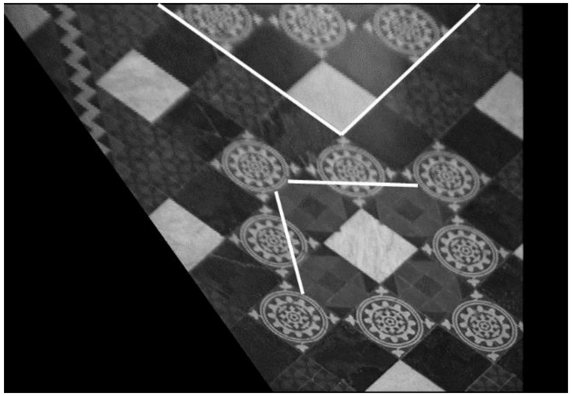
$$\mathbf{C}_\infty^*{}' = \mathbf{U}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{U}^\mathsf{T} \qquad \mathbf{H} = \mathbf{U}$$

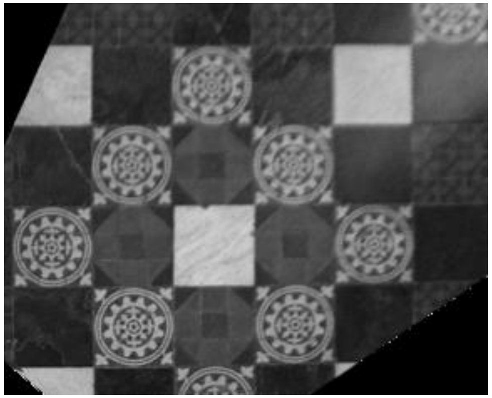
Metric from Affine

Suppose an image has been affinely rectified (**v**=0)

$$\begin{pmatrix} l_1' & l_2' & l_3' \end{pmatrix} \begin{bmatrix} \mathbf{KK}^\mathsf{T} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} m_1' \\ m_2' \\ m_3' \end{pmatrix} = 0$$

$$\left(l_1'm_1', l_1'm_2' + l_2'm_1', l_2'm_2'\right)\left(k_{11}^2 + k_{12}^2, k_{11}k_{12}, k_{22}^2\right)^\mathsf{T} = 0$$





Metric from Projective

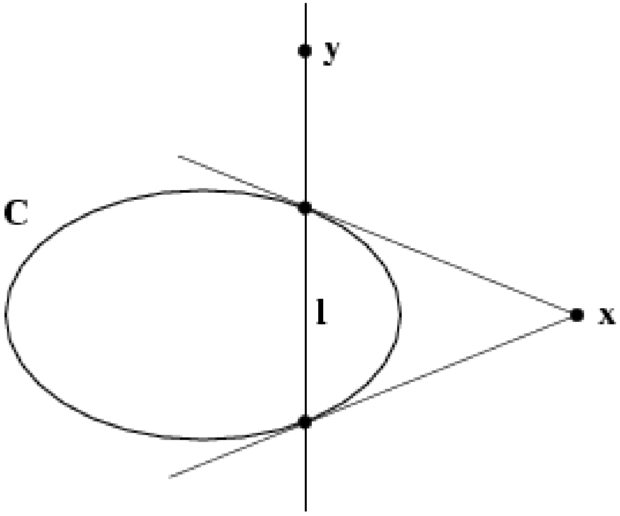
$$\mathrm{l}^{\mathsf{T}}\,\mathbf{C}_{\infty}^{*}\,\mathrm{m}=0 \qquad \begin{pmatrix} l_1' & l_2' & l_3' \end{pmatrix}\begin{bmatrix} \mathbf{K}\mathbf{K}^{\mathsf{T}} & \mathbf{K}^{\mathsf{T}}\mathrm{v} \\ \mathrm{v}^{\mathsf{T}}\mathbf{K} & \mathrm{v}^{\mathsf{T}}\mathrm{v} \end{bmatrix}\begin{pmatrix} m_1' \\ m_2' \\ m_3' \end{pmatrix}=0$$

$$\left(l_1'm_1', 0.5\left(l_1'm_2'+l_2'm_1'\right), l_2'm_2', 0.5\left(l_1'm_3'+l_3'm_1'\right), 0.5\left(l_2'm_3'+l_3'm_2'\right), l_3'm_3'\right)\mathrm{c}=0$$

$$\mathbf{c}=(a,b,c,d,e,f)^{\mathrm{T}}$$

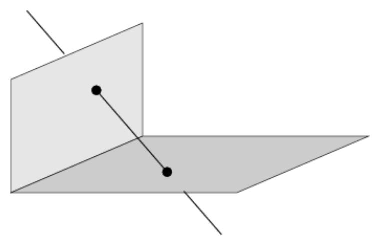
Pole-polar Relationship

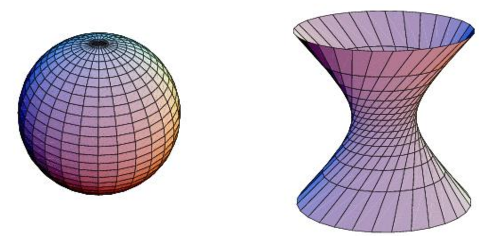
The polar line $\mathrm{l}=\mathbf{C}\mathrm{x}$ of the point x with respect to the conic **C** intersects the conic in two points. The two lines tangent to **C** at these points intersect at x



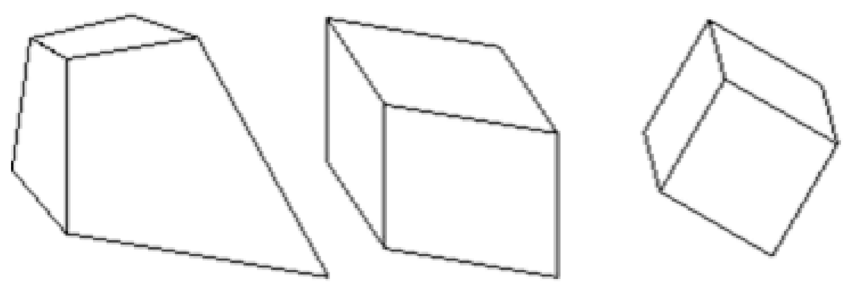
Projective 3D Geometry

- Points, lines, planes and quadrics

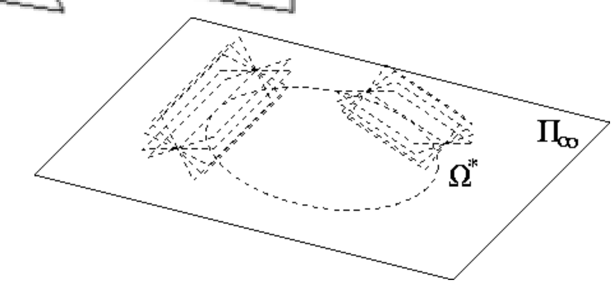




- Transformations



- Π_∞, ω_∞ and Ω_∞



3D Points

3D point

$\left(X,Y,Z\right)^{\mathsf{T}}$ in $\mathbf{R}^3$

$\mathrm{X}=\left(X_1,X_2,X_3,X_4\right)^{\mathsf{T}}$ in $\mathbf{P}^3$

$$\mathrm{X}=\left(\frac{X_1}{X_4},\frac{X_2}{X_4},\frac{X_3}{X_4},1\right)^{\mathsf{T}}=\left(X,Y,Z,1\right)^{\mathsf{T}} \qquad \left(X_4\neq 0\right)$$

projective transformation

$$\mathrm{X}'=\mathbf{H}\,\mathrm{X} \quad \text{(4x4-1=15 dof)}$$

Dual: points ↔ planes, lines ↔ lines

Planes

3D plane

$$\pi_1 X + \pi_2 Y + \pi_3 Z + \pi_4 = 0$$

$$\pi_1 X_1 + \pi_2 X_2 + \pi_3 X_3 + \pi_4 X_4 = 0$$

$$\pi^\mathsf{T} \mathrm{X} = 0$$

Transformation

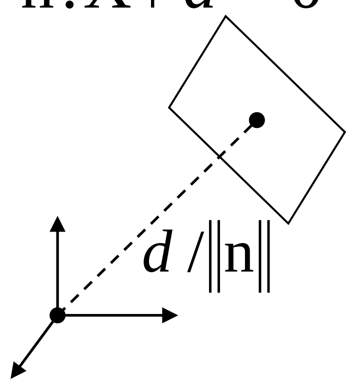
$$\mathrm{X}' = \mathbf{H}\,\mathrm{X}$$

$$\pi' = \mathbf{H}^{-\mathsf{T}}\,\pi$$

Euclidean representation

$$\mathrm{n}.\tilde{\mathrm{X}} + d = 0 \qquad \mathrm{n} = (\pi_1, \pi_2, \pi_3)^\mathsf{T} \qquad \tilde{\mathrm{X}} = (X, Y, Z)^\mathsf{T}$$

$$\pi_4 = d \qquad X_4 = 1$$



Planes from Points

Solve π from $\mathrm{X}_1^{\mathsf{T}}\pi = 0, \mathrm{X}_2^{\mathsf{T}}\pi = 0$ and $\mathrm{X}_3^{\mathsf{T}}\pi = 0$

$$\begin{bmatrix} \mathrm{X}_1^{\mathsf{T}} \\ \mathrm{X}_2^{\mathsf{T}} \\ \mathrm{X}_3^{\mathsf{T}} \end{bmatrix} \pi = 0$$

(solve $\boldsymbol{\pi}$ as right nullspace of $\begin{bmatrix} \mathrm{X}_1^{\mathsf{T}} \\ \mathrm{X}_2^{\mathsf{T}} \\ \mathrm{X}_3^{\mathsf{T}} \end{bmatrix}$)

Or implicitly from coplanarity condition

$$\det \begin{bmatrix} X_1 & (X_1)_1 & (X_2)_1 & (X_3)_1 \\ X_2 & (X_1)_2 & (X_2)_2 & (X_3)_2 \\ X_3 & (X_1)_3 & (X_2)_3 & (X_3)_3 \\ X_4 & (X_1)_4 & (X_2)_4 & (X_3)_4 \end{bmatrix} = 0$$

$$X_1 D_{234} - X_2 D_{134} + X_3 D_{124} - X_4 D_{123} = 0$$

$$\pi = \left(D_{234}, -D_{134}, D_{124}, -D_{123}\right)^{\mathsf{T}}$$

Points from Planes

Solve X from $\pi_1^{\mathsf{T}}\mathrm{X} = 0, \pi_2^{\mathsf{T}}\mathrm{X} = 0$ and $\pi_3^{\mathsf{T}}\mathrm{X} = 0$

$$\begin{bmatrix} \pi_1^{\mathsf{T}} \\ \pi_2^{\mathsf{T}} \\ \pi_3^{\mathsf{T}} \end{bmatrix} \mathrm{X} = 0$$ (solve X as right nullspace of $\begin{bmatrix} \pi_1^{\mathsf{T}} \\ \pi_2^{\mathsf{T}} \\ \pi_3^{\mathsf{T}} \end{bmatrix}$)

Points and Planes

- Projective transformation

  Under the point transformation $\boldsymbol{X}' = H\boldsymbol{X}$, a plane transforms as $\boldsymbol{\pi}' = H^{-T}\boldsymbol{\pi}$

- Parametrized points on a plane

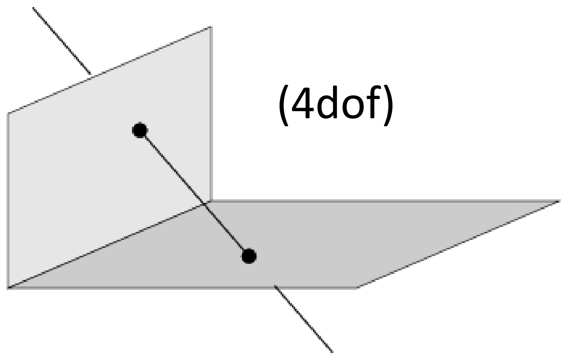
  Representing a plane $\pi = (a, b, c, d)^{\mathsf{T}}$ by its span

  $\mathrm{X} = \mathbf{M}\,\mathrm{x}$ x is a 3-vector parameter (a point on the projective plane)

  $\pi^{\mathsf{T}}\,\mathbf{M} = 0$

  M is not unique $\mathbf{M} = \begin{bmatrix} \mathrm{p} \\ \mathrm{I}_{3\mathrm{x}3} \end{bmatrix}$ $p = \left(-\dfrac{b}{a}, -\dfrac{c}{a}, -\dfrac{d}{a}\right)$

Lines



(4dof)

Defined as the join of two points A, B

$$\mathrm{W} = \begin{bmatrix} \mathrm{A}^{\mathsf{T}} \\ \mathrm{B}^{\mathsf{T}} \end{bmatrix} \qquad \lambda\mathrm{A} + \mu\mathrm{B}$$

(Dual) Defined as the intersection of two planes P, Q

$$\mathrm{W}^{*} = \begin{bmatrix} \mathrm{P}^{\mathsf{T}} \\ \mathrm{Q}^{\mathsf{T}} \end{bmatrix} \qquad \lambda\mathrm{P} + \mu\mathrm{Q}$$

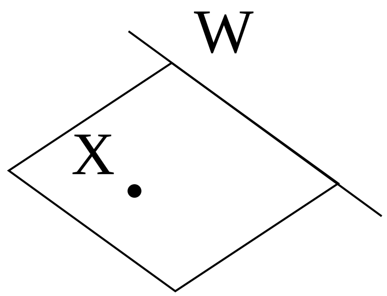
$$\mathrm{W}^{*}\mathrm{W}^{\mathsf{T}} = \mathrm{W}\mathrm{W}^{*\mathsf{T}} = 0_{2\times 2}$$

Example: X-axis

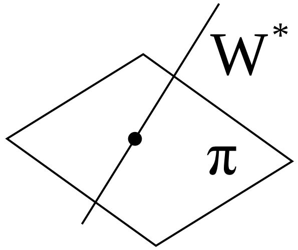
$$\mathrm{W} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \mathrm{W}^{*} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Points, Lines and Planes

$$\mathbf{M} = \begin{bmatrix} W \\ X^{\mathsf{T}} \end{bmatrix} \qquad \mathbf{M}\pi = 0$$



$$\mathbf{M} = \begin{bmatrix} W^{*} \\ \pi^{\mathsf{T}} \end{bmatrix} \qquad \mathbf{M}X = 0$$



Plücker Matrices

Plücker matrix (4x4 skew-symmetric homogeneous matrix)

$$l_{ij} = A_i B_j - B_i A_j$$

$$\mathrm{L} = \mathrm{A}\mathrm{B}^\mathsf{T} - \mathrm{B}\mathrm{A}^\mathsf{T}$$

1. L has rank 2 $\quad \mathrm{L}\mathrm{W}^{*\mathsf{T}} = 0_{4\times 2}$
2. 4dof
3. generalization of $\mathrm{l} = \mathrm{x} \times \mathrm{y}$
4. L independent of choice A and B
5. Transformation $\quad \mathrm{L}' = \mathrm{H}\mathrm{L}\mathrm{H}^\mathsf{T}$

Example: X-axis

$$\mathrm{L} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^\mathsf{T} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Plücker Matrices

Dual Plücker matrix L^*

$$\mathrm{L}^* = \mathrm{P}\mathrm{Q}^{\mathsf{T}} - \mathrm{Q}\mathrm{P}^{\mathsf{T}}$$

$$\mathrm{L}^{*'} = \mathrm{H}^{-\mathsf{T}}\mathrm{L}\mathrm{H}^{-1}$$

Correspondence

$$l_{12} : l_{13} : l_{14} : l_{23} : l_{42} : l_{34} = l^*_{34} : l^*_{42} : l^*_{23} : l^*_{14} : l^*_{13} : l^*_{12}$$

Join and incidence

$\pi = \mathrm{L}^*\mathrm{X}$ (plane through point and line)

$\mathrm{L}^*\mathrm{X} = 0$ (point on line)

$\mathrm{X} = \mathrm{L}\pi$ (intersection point of plane and line)

$\mathrm{L}\pi = 0$ (line in plane)

$[\mathrm{L}_1, \mathrm{L}_2, \ldots]\pi = 0$ (coplanar lines)

Quadrics and Dual Quadrics

$\mathrm{X}^{\mathsf{T}}\mathrm{QX}=0$ (Q : 4x4 symmetric matrix)

$$\mathrm{Q}=\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \bullet \end{bmatrix}$$

1. 9 d.o.f.
2. in general 9 points define quadric
3. det Q=0 $\boxed{\leftrightarrow}$ degenerate quadric
4. Polar plane $\pi=\mathrm{QX}$
5. (plane ∩ quadric)=conic $\mathrm{C}=\mathrm{M}^{\mathsf{T}}\mathrm{QM}$ $\pi:\mathrm{X}=\mathrm{Mx}$
6. transformation $\mathrm{Q}'=\mathrm{H}^{-\mathsf{T}}\mathrm{QH}^{-1}$

Q^*: dual quadric, equations on planes

$\pi^{\mathsf{T}}\mathrm{Q}^*\pi=0$

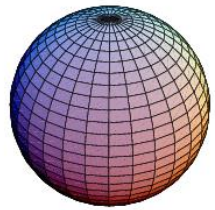
1. relation to quadric $\mathrm{Q}^*=\mathrm{Q}^{-1}$ (non-degenerate)
2. transformation $\mathrm{Q}'^*=\mathrm{HQ}^*\mathrm{H}^{\mathsf{T}}$

Quadric Classification

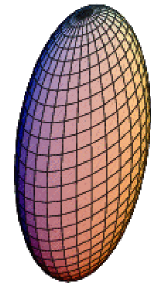
Rank	Sign.	Diagonal	Equation	Realization
4	4	(1,1,1,1)	$X^2+Y^2+Z^2+1=0$	No real points
	2	(1,1,1,-1)	$X^2+Y^2+Z^2=1$	Sphere
	0	(1,1,-1,-1)	$X^2+Y^2=Z^2+1$	Hyperboloid (1S)
3	3	(1,1,1,0)	$X^2+Y^2+Z^2=0$	Single point
	1	(1,1,-1,0)	$X^2+Y^2=Z^2$	Cone
2	2	(1,1,0,0)	$X^2+Y^2=0$	Single line
	0	(1,-1,0,0)	$X^2=Y^2$	Two planes
1	1	(1,0,0,0)	$X^2=0$	Single plane

Quadric Classification

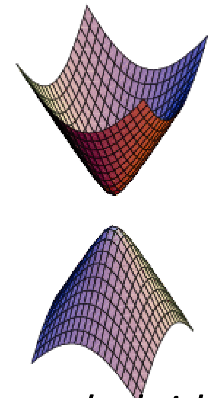
Projectively equivalent to *sphere:*



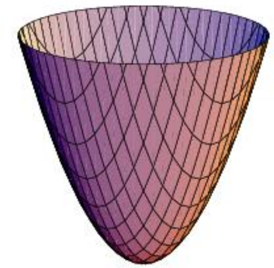
sphere



ellipsoid

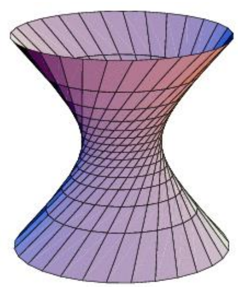


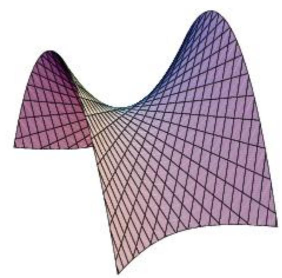
hyperboloid of two sheets



paraboloid

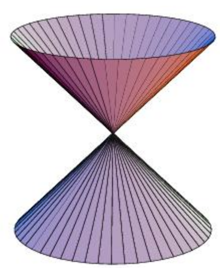
Ruled quadrics: (contain straight line)



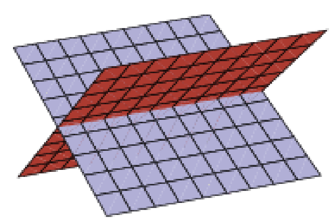


hyperboloids of one sheet

Degenerate ruled quadrics:



cone



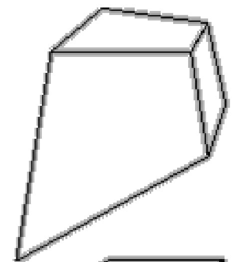
two planes

Hierarchy of Transformations

Invariant Properties

Projective
15dof

$$\begin{bmatrix} A & t \\ v^T & v \end{bmatrix}$$

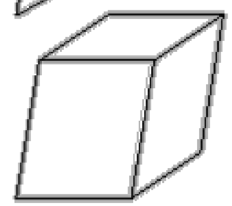


Intersection and tangency

5 for affine scaling

Affine
12dof

$$\begin{bmatrix} A & t \\ 0^T & 1 \end{bmatrix}$$

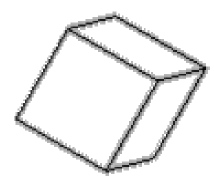


Parallellism of planes,
Volume ratios, centroids,
The plane at infinity π_∞

3 for rotation
3 for translation
1 for isotropic scaling

Similarity
7dof

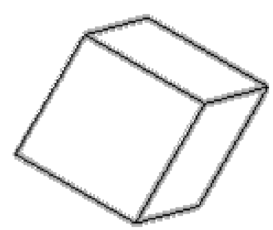
$$\begin{bmatrix} sR & t \\ 0^T & 1 \end{bmatrix}$$



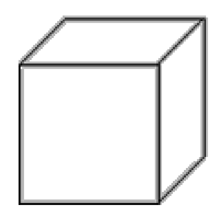
The absolute conic Ω_∞

Euclidean
6dof

$$\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$$

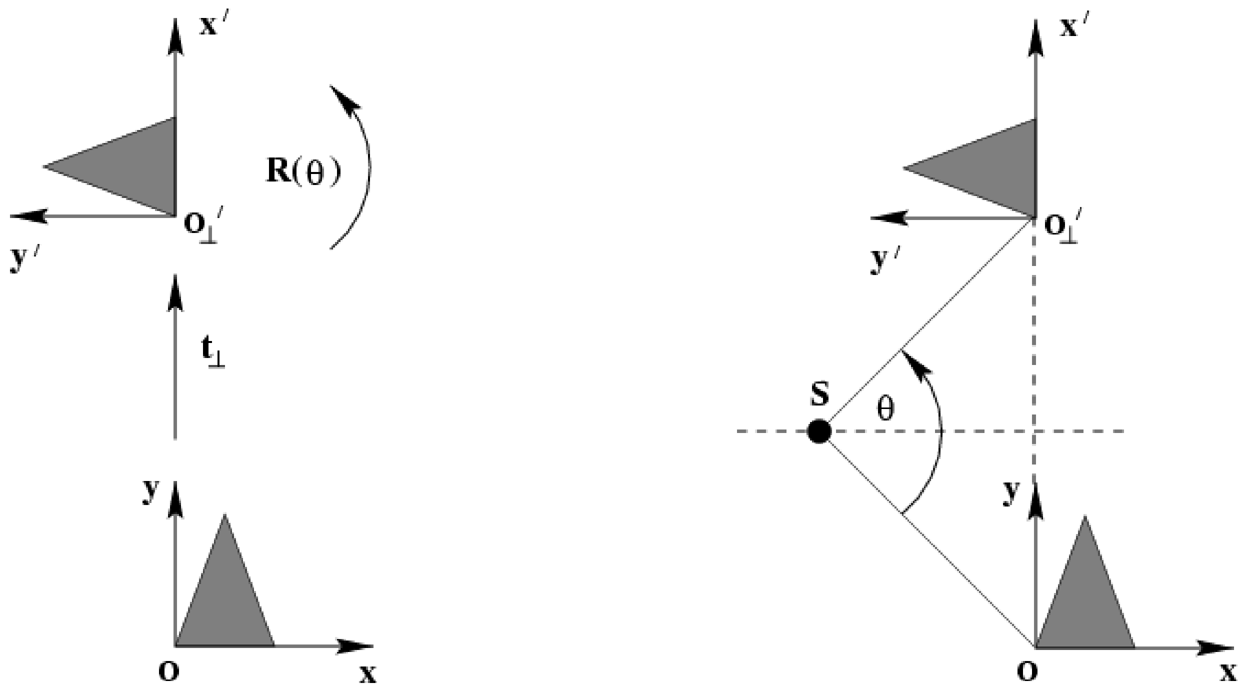


Volume



Screw Decomposition

Any particular translation and rotation is equivalent to a rotation about a screw axis and a translation along the screw axis.



Screw Decomposition

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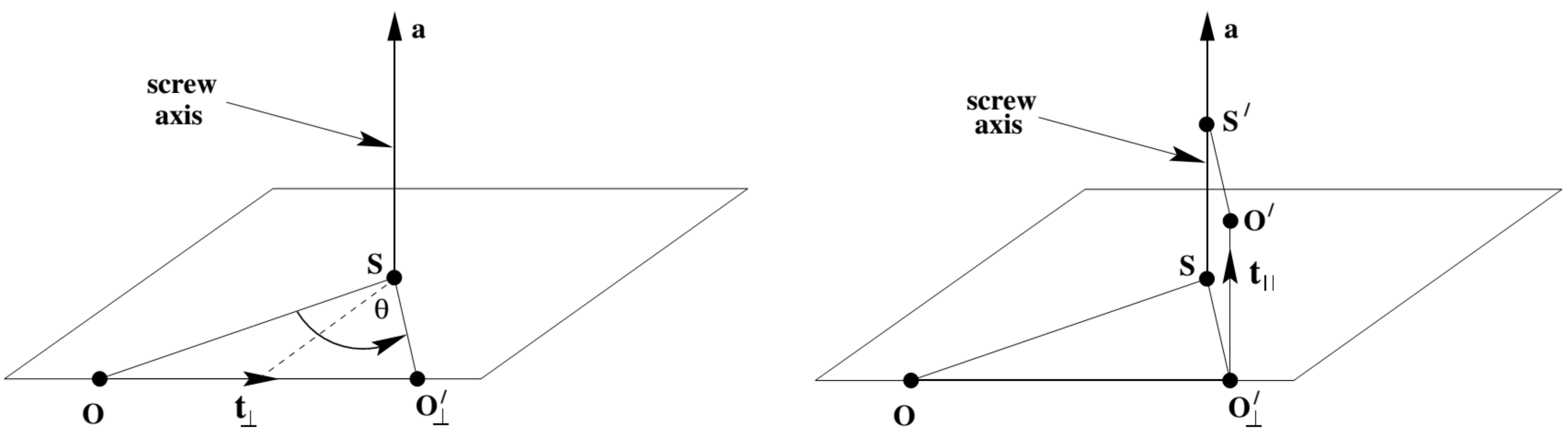

screw axis // rotation axis

$$t = t_{//} + t_{\perp}$$

The Plane at Infinity

$$\pi'_\infty = \mathbf{H}_A^{-\mathsf{T}} \pi_\infty = \begin{bmatrix} \mathbf{A}^{-\mathsf{T}} & 0 \\ -\mathbf{A}\,\mathrm{t} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \pi_\infty$$

The plane at infinity π_∞ is a fixed plane under a projective transformation H iff H is an affinity

1. canical position $\pi_\infty = (0,0,0,1)^\mathsf{T}$
2. contains directions $\mathrm{D} = (X_1, X_2, X_3, 0)^\mathsf{T}$
3. two planes are parallel $\Leftrightarrow$ line of intersection in π_∞
4. line // line (or plane) $\Leftrightarrow$ point of intersection in π_∞