Two-View Geometry: Epipolar Geometry and the Fundamental Matrix

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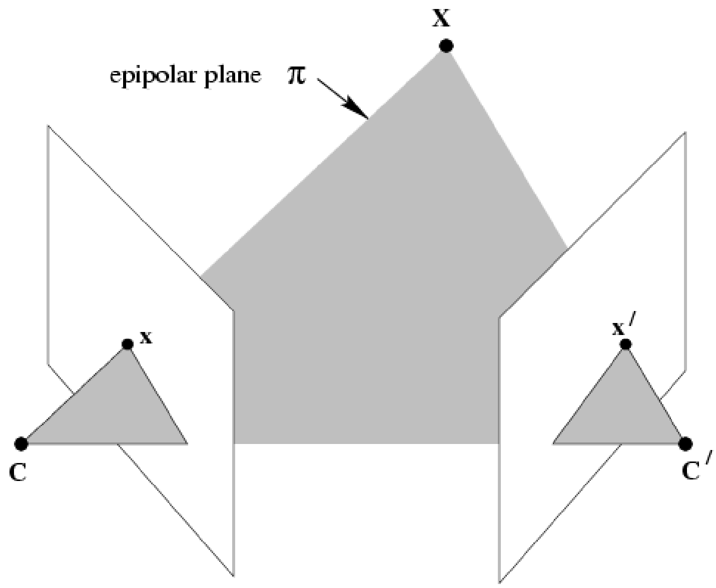
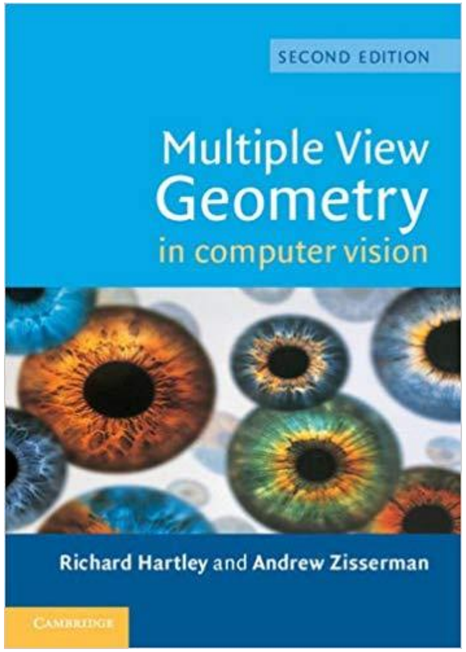
Outline

- Epipolar geometry and the fundamental matrix



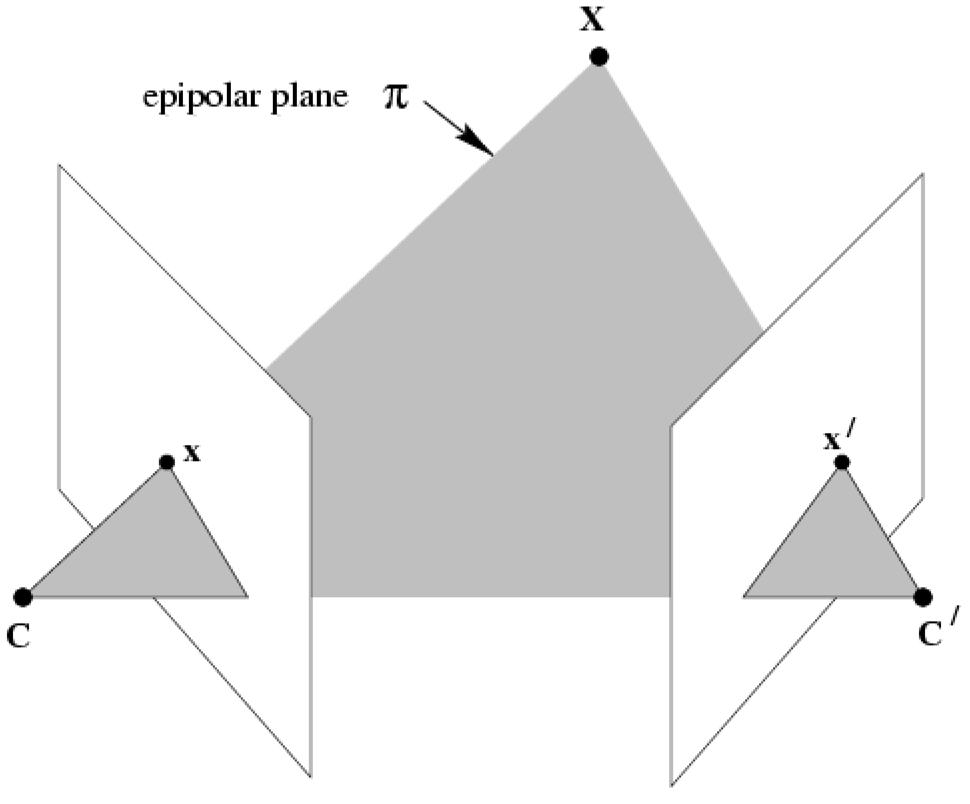


[Slides credit: Marc Pollefeys]

Three Questions

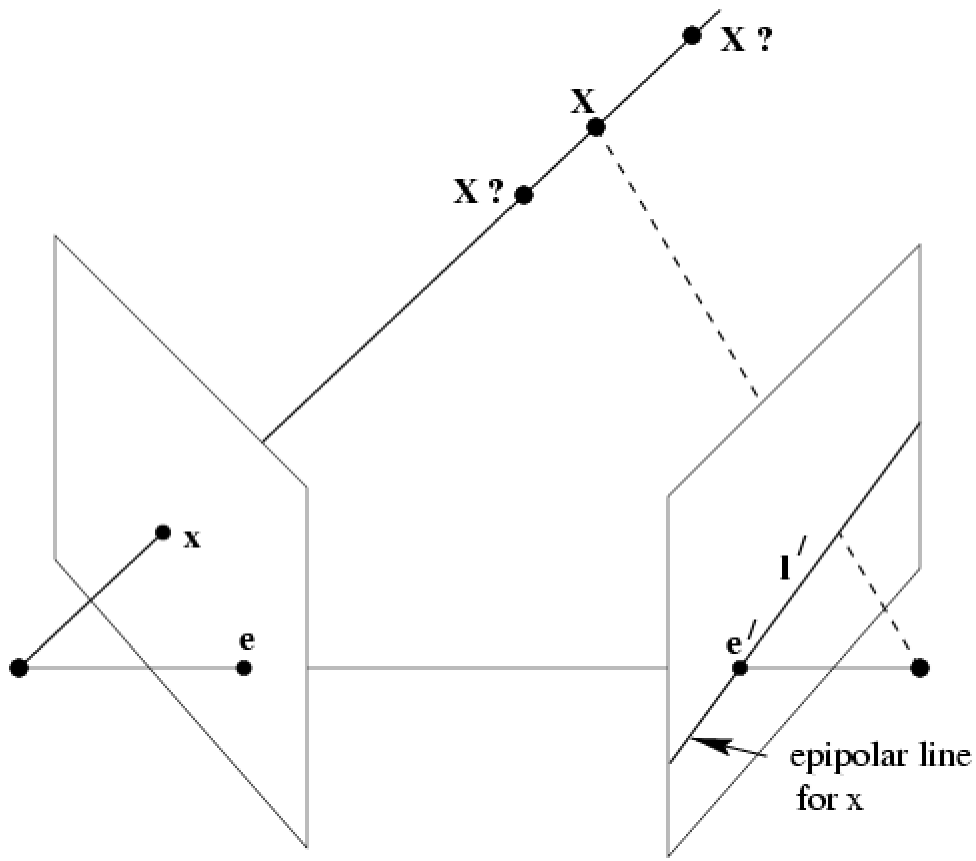
- **Correspondence geometry:** Given an image point x in the first view, how does this constrain the position of the corresponding point x' in the second image?
- **Camera geometry (motion):** Given a set of corresponding image points $\{x_i \leftrightarrow x'_i\}$, i=1,…,n, what are the cameras P and P' for the two views?
- **Scene geometry (structure):** Given corresponding image points $x_i \leftrightarrow x'_i$ and cameras P, P', what is the position of (their pre-image) X in space?

The Epipolar Geometry



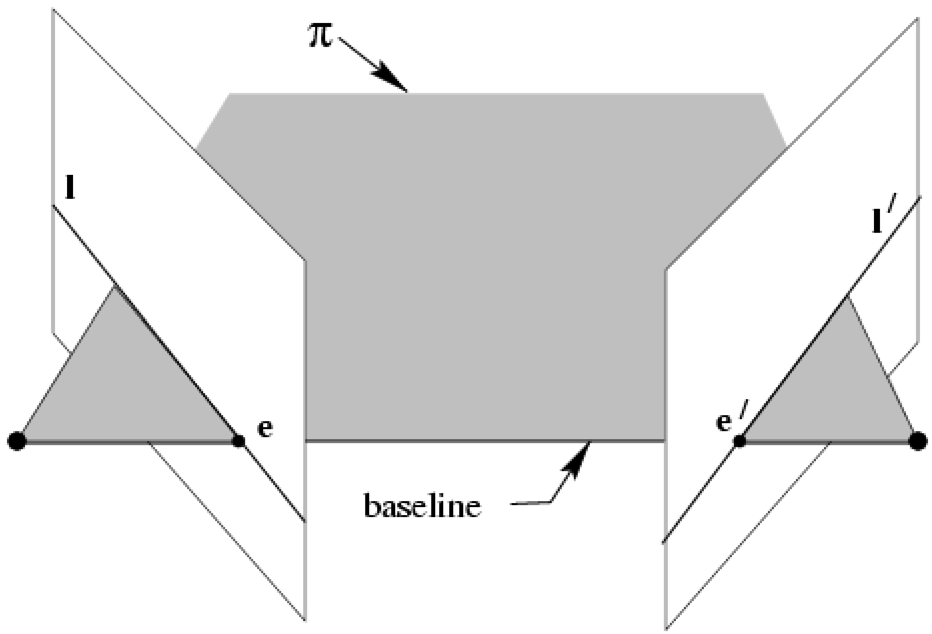
C,C',x,x' and X are coplanar

The Epipolar Geometry



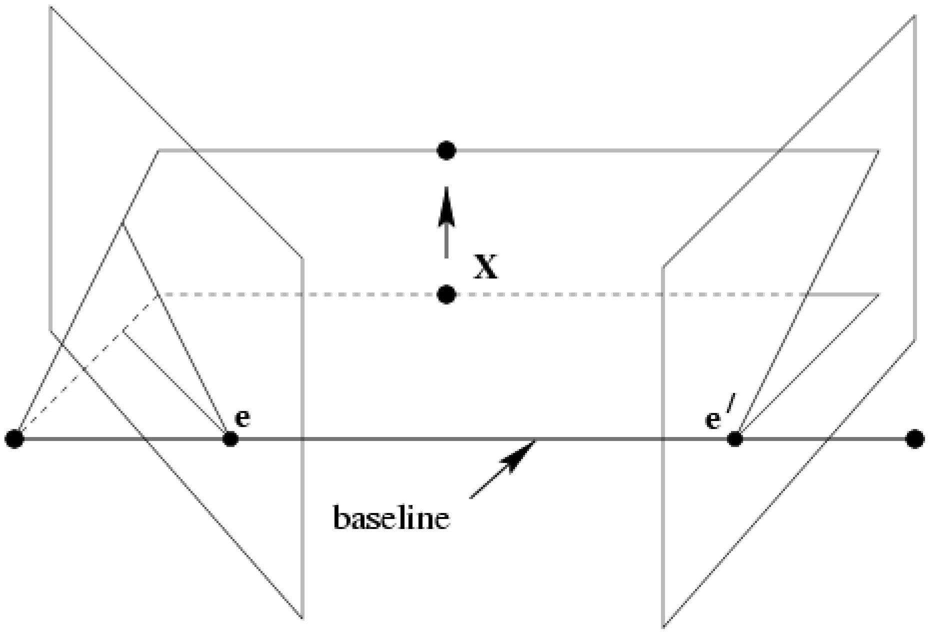
What if only C,C',x are known?

The Epipolar Geometry



All points on π project on l and l'

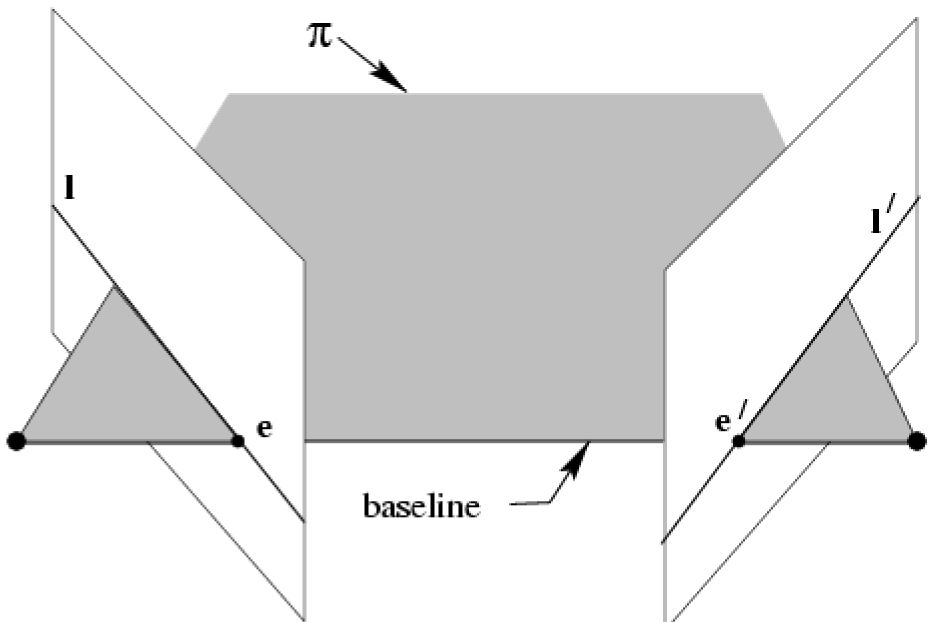
The Epipolar Geometry



Family of planes π and lines l and l'
Intersection in e and e'

The Epipolar Geometry

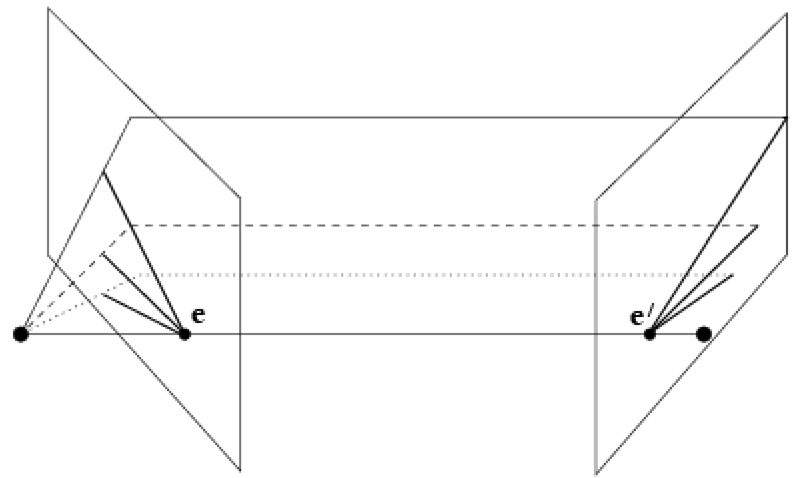
Epipoles e,e’
= intersection of baseline with image plane
= projection of projection center in other image
= vanishing point of camera motion direction

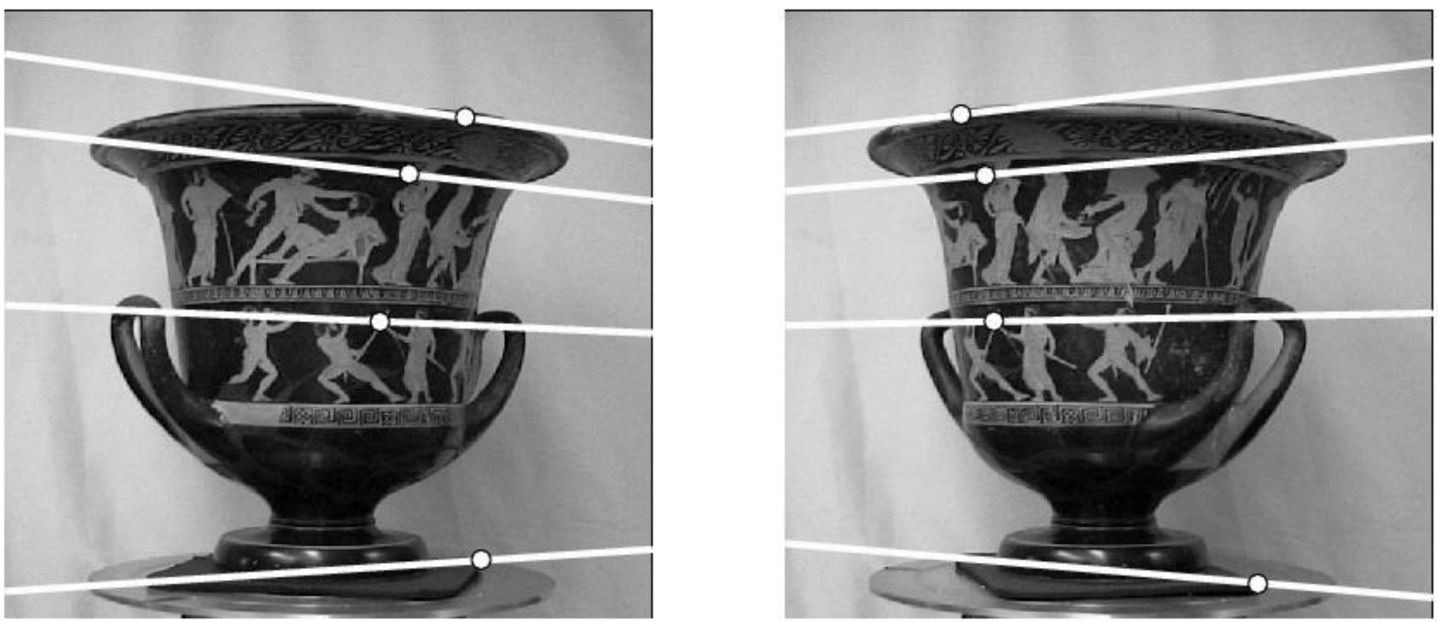


an epipolar plane = plane containing baseline (1-D family)

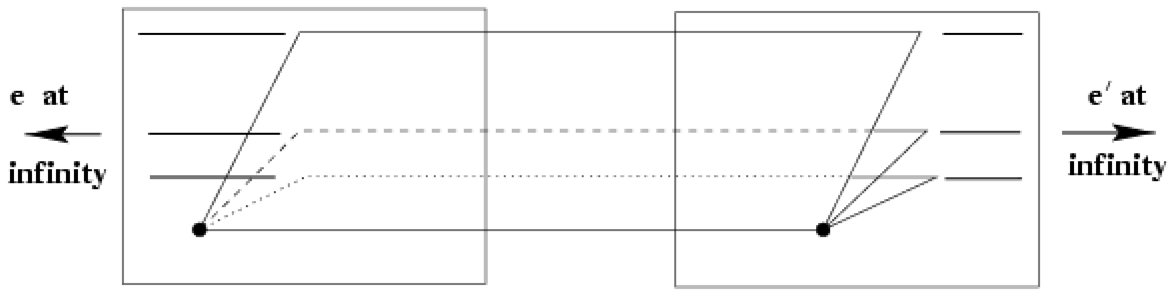
an epipolar line = intersection of epipolar plane with image
(always come in corresponding pairs)

Example: Converging Cameras

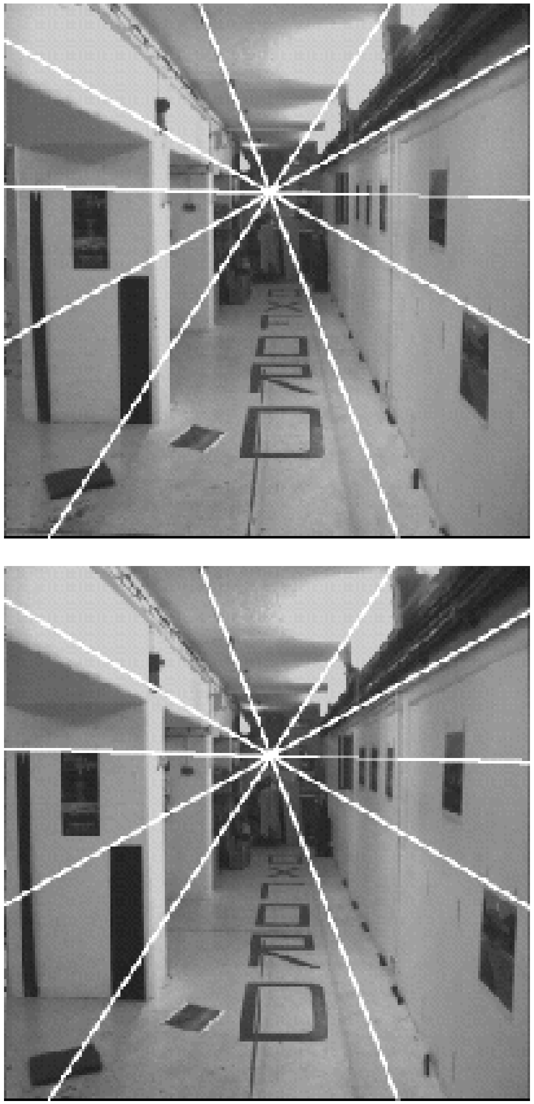


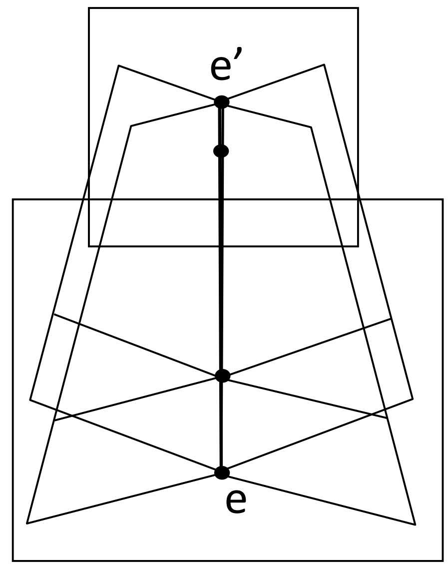


Example: Motion Parallel with Image Plane



Example: Forward Motion





The Fundamental Matrix F

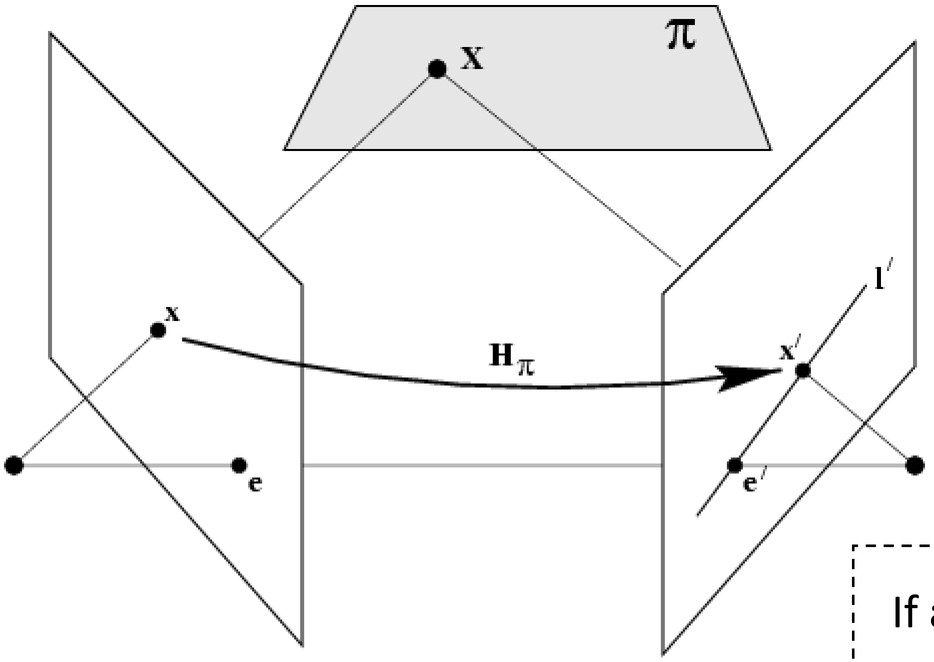
Algebraic representation of epipolar geometry

$$x \mapsto l'$$

we will see that mapping is (singular) correlation
(i.e. projective mapping from points to lines)
represented by the fundamental matrix F

The Fundamental Matrix F

geometric derivation



$$x' = H_\pi x$$

$$l' = e' \times x' = [e']_\times H_\pi x = Fx$$

If $\mathbf{a}=(a_1, a_2, a_3)^T$

$$[\boldsymbol{a}]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

mapping from 2-D to 1-D family (rank 2)

The Fundamental Matrix F

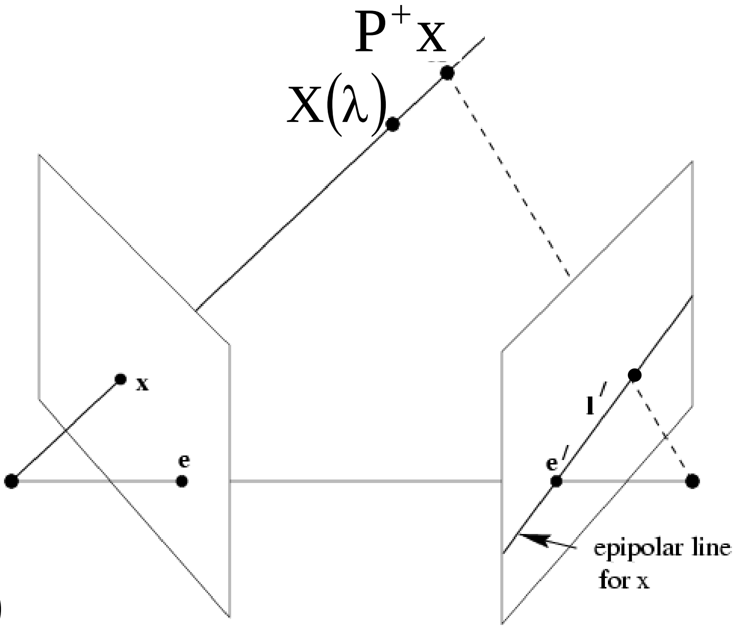
algebraic derivation

$$X(\lambda) = P^{+}x + \lambda C$$

$$\left(P^{+}P = I\right)$$

$$l = \underbrace{P'C}_{e'} \times P'P^{+}x$$

$$F = [e']_{\times} P'P^{+}$$



(note: doesn't work for C=C' $\Rightarrow$ F=0)

The Fundamental Matrix F

correspondence condition

The fundamental matrix satisfies the condition that for any pair of corresponding points $x \leftrightarrow x'$ in the two images

$$x'^{T} F x = 0 \qquad \left(x'^{T} l' = 0\right)$$

The Fundamental Matrix F

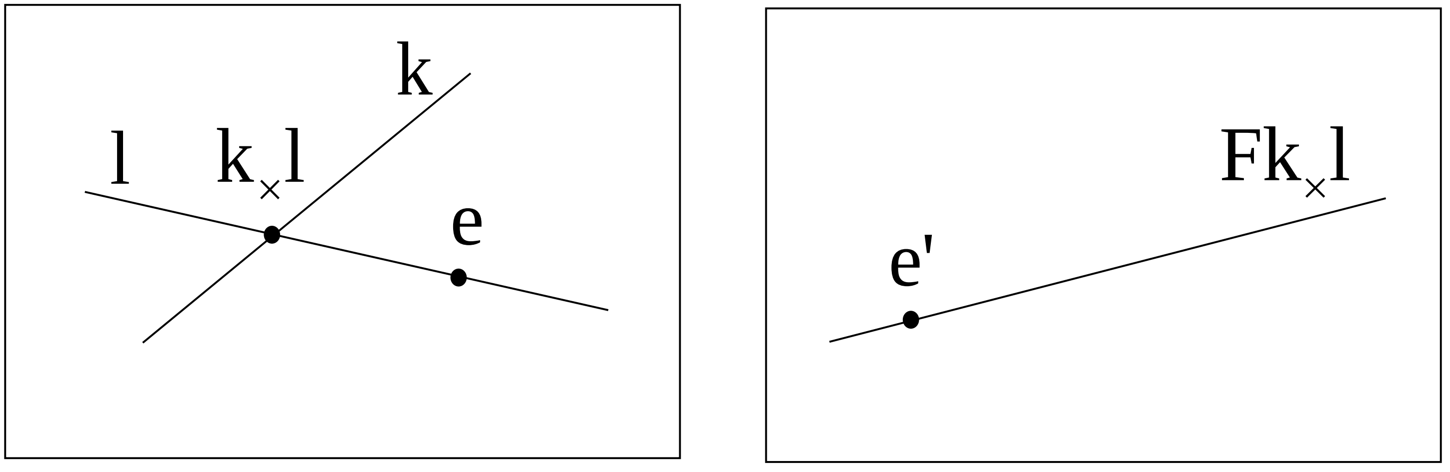
F is the unique 3x3 rank 2 matrix that satisfies
$x'^T F x = 0$ for all $x \leftrightarrow x'$

- **(i) Transpose:** if F is fundamental matrix for (P,P'), then F^T is fundamental matrix for (P',P)
- **(ii) Epipolar lines:** $l' = Fx$ & $l = F^T x'$
- **(iii) Epipoles:** on all epipolar lines, thus $e'^T F x = 0, \forall x \Rightarrow e'^T F = 0$, similarly $Fe = 0$
- **(iv) F** has 7 d.o.f. , i.e. 3x3-1(homogeneous)-1(rank2)
- **(v) F** is a correlation, projective mapping from a point x to a line $l' = Fx$ (not a proper correlation, i.e. not invertible)

The Epipolar Line Geometry

l,l' epipolar lines, k line not through e

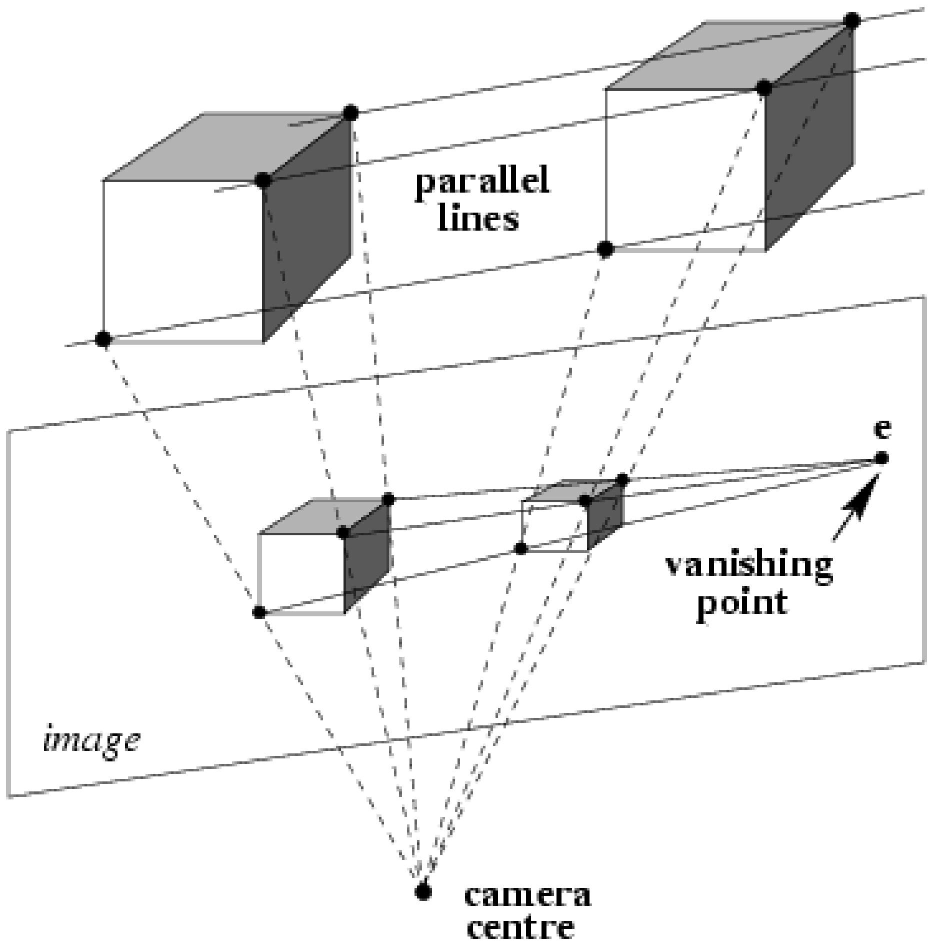
$\Rightarrow$ $l' = F[k]_x l$ and symmetrically $l = F^T[k']_x l'$



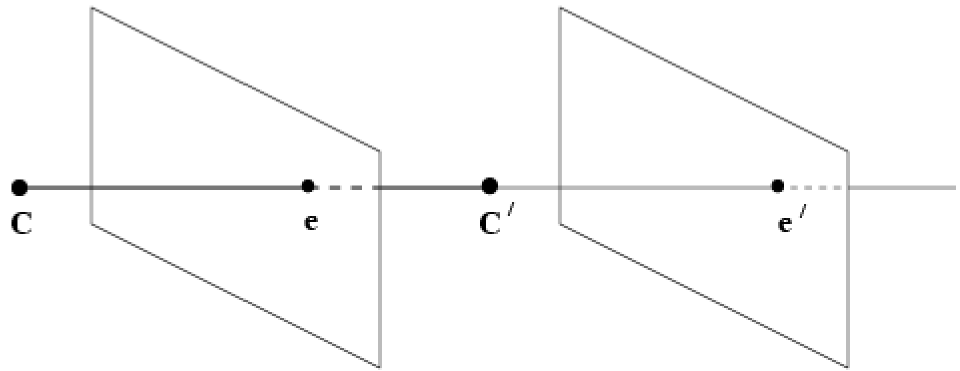
(pick k=e, since $e^T e \neq 0$)

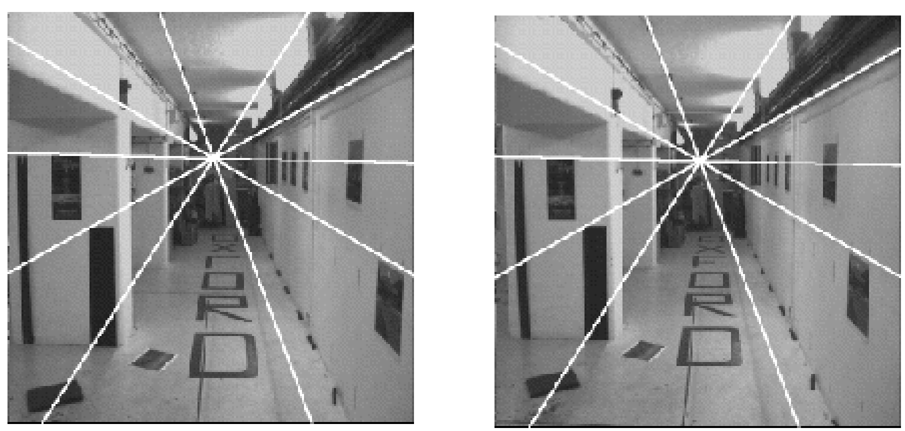
$$l' = F[e]_\times l \qquad l = F^T[e']_\times l'$$

Fundamental Matrix for Pure Translation



Fundamental Matrix for Pure Translation





Fundamental Matrix for Pure Translation

$$F = [e']_\times H_\infty = [e']_\times \qquad (H_\infty = K^{-1}RK)$$

example:

P=K[I | 0], P'=K[I | t]

Translation is parallel to the x-axis

$$e' = (1,0,0)^T \qquad F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}_\times$$

$$x'^T Fx = 0 \Leftrightarrow y = y'$$

Fundamental Matrix for Pure Translation

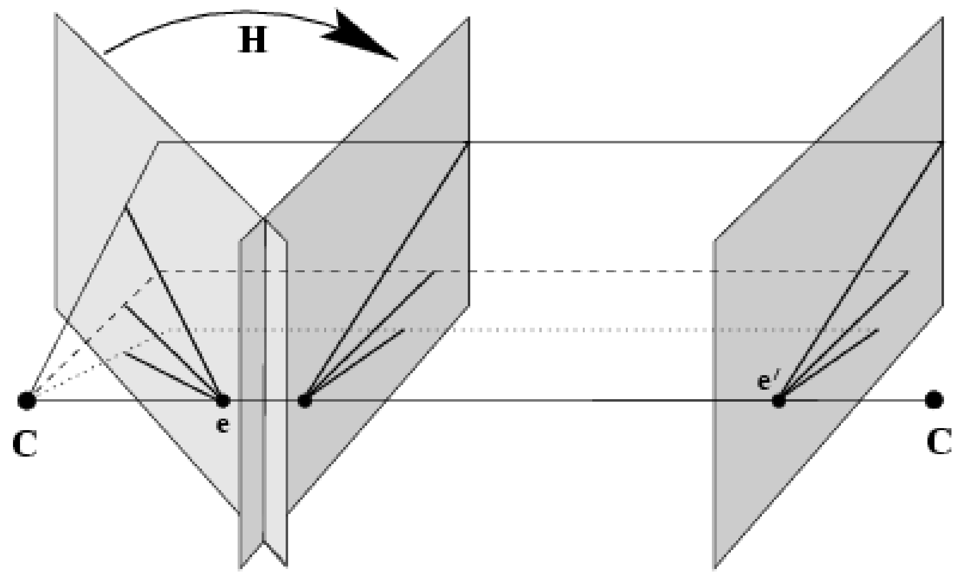
$$\mathrm{x} = \mathrm{PX} = \mathrm{K[I\,|\,0]X} \qquad (X,Y,Z)^{\mathsf{T}} = \mathrm{K}^{-1}\mathrm{x}/Z$$

$$\mathrm{x}' = \mathrm{P'X} = \mathrm{K[I\,|\,t]}\begin{bmatrix}\mathrm{K}^{-1}\mathrm{x} \\ Z\end{bmatrix} \qquad \mathrm{x}' = \mathrm{x} + \mathrm{Kt}/Z$$

motion starts at x and moves towards e, faster depending on *Z*

pure translation: F only 2 d.o.f., $x^{T}[e]_{x}x=0 \Rightarrow$ auto-epipolar

General Motion



$$x'^{T}[e']_{\times}Hx = 0$$

$$x'^{T}[e']_{\times}\hat{x} = 0$$

$$x' = K'RK^{-1}x + K't/Z$$

Projective Transformation and Invariance

Derivation based purely on projective concepts

$$\hat{x} = Hx,\ \hat{x}' = H'x' \Rightarrow \hat{F} = H'^{-T} F H^{-1}$$

F invariant to transformations of projective 3-space

$$x = PX = (PH)(H^{-1}X) = \hat{P}\hat{X}$$

$$x' = P'X = (P'H)(H^{-1}X) = \hat{P}'\hat{X}$$

Same matching point!

$$(P, P') \mapsto F$$ unique

$$F \mapsto (P, P')$$ not unique

canonical form

$$P = [I \mid 0]$$
$$P' = [M \mid m]$$

$$F = [m]_\times M$$

Projective Ambiguity of Cameras Given F

previous slide: at least projective ambiguity
this slide: not more!

Show that if F is same for (P,P') and ($\tilde{P}$,$\tilde{P}$'),
there exists a projective transformation H so that
$\tilde{P}$=PH and $\tilde{P}$'=P'H

$$P = [I \mid 0] \quad P' = [A \mid a] \quad \tilde{P} = [I \mid 0] \quad \tilde{P}' = [\tilde{A} \mid \tilde{a}]$$

$$F = [a]_\times A = [\tilde{a}]_\times \tilde{A}$$

lemma: $\tilde{a} = ka \quad \tilde{A} = k^{-1}\left(A + av^T\right)$

$$aF = a[a]_\times A = 0 = \tilde{a}F \overset{\text{rank 2}}{\Longrightarrow} \tilde{a} = ka$$

$$[a]_\times A = [\tilde{a}]_\times \tilde{A} \Rightarrow [a]_\times \left(k\tilde{A} - A\right) = 0 \Rightarrow \left(k\tilde{A} - A\right) = av^T$$

$$H = \begin{bmatrix} k^{-1}I & 0 \\ k^{-1}v^T & k \end{bmatrix}$$

$$P'H = [A \mid a]\begin{bmatrix} k^{-1}I & 0 \\ k^{-1}v^T & k \end{bmatrix} = [k^{-1}\left(A - av^T\right) \mid ka] = \tilde{P}'$$

Canonical Cameras Given F

F matrix corresponds to P,P' iff $P'^T FP$ is skew-symmetric

$$\left(X^T P'^T FPX = 0, \forall X\right)$$

F matrix, S skew-symmetric matrix

$$P = [I \mid 0] \quad P' = [SF \mid e']$$ (fund.matrix=F)

$$\left([SF \mid e']^T F[I \mid 0] = \begin{bmatrix} F^T S^T F & 0 \\ e'^T F & 0 \end{bmatrix} = \begin{bmatrix} F^T S^T F & 0 \\ 0 & 0 \end{bmatrix} \right)$$

Possible choice:

$$P = [I \mid 0] \quad P' = [[e']_\times F \mid e']$$

Canonical representation:

$$P = [I \mid 0] \quad P' = [[e']_\times F + e' v^T \mid \lambda e']$$

The Essential Matrix

≡fundamental matrix for calibrated cameras (remove K)

$$E = [t]_\times R = R[R^T t]_\times$$

$$\hat{x}'^T E \hat{x} = 0 \qquad \left(\hat{x} = K^{-1}x;\ \hat{x}' = K^{-1}x'\right)$$

$$E = K'^T F K$$

5 d.o.f. (3 for R; 2 for t up to scale)

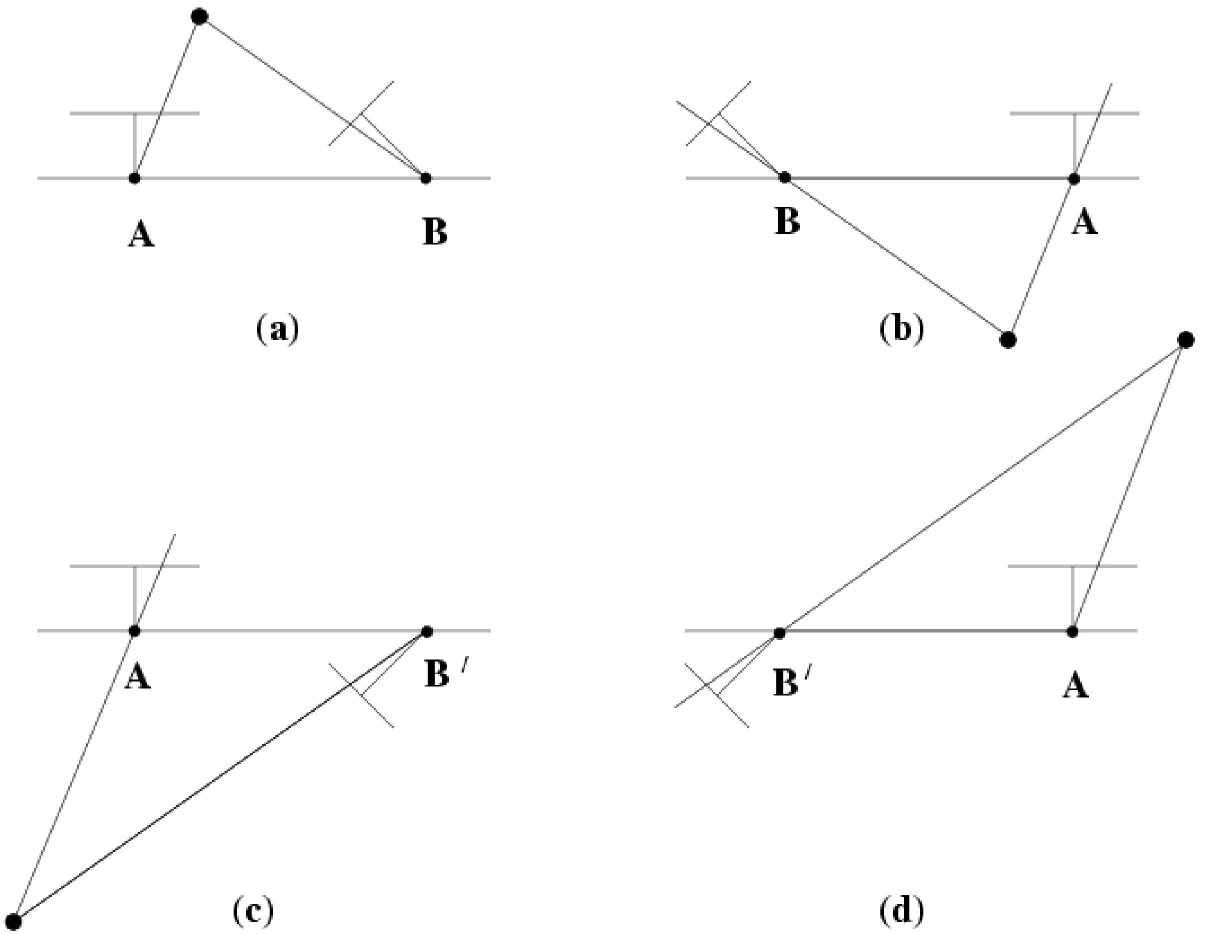
E is essential matrix if and only if
two singularvalues are equal (and third=0)

$$E = U\mathrm{diag}(1,1,0)V^T$$

Given E, P=[I|0], there are 4 possible choices for the second camera matrix P’

$$P' = [UWV^T \mid +\mathbf{u}_3] \text{ or } [UWV^T \mid -\mathbf{u}_3] \text{ or } [UW^TV^T \mid +\mathbf{u}_3] \text{ or } [UW^TV^T \mid -\mathbf{u}_3]$$

Four Possible Reconstructions from E



(only one solution where points is in front of both cameras)